LESSON 2

NUMBER PATTERNS

Revision

First we need you to recall the following from Grade 10:

Example

Example 1

The general term (T_n) of a sequence is given by $T_n = 6n + 2$. Find the first four terms.

Solution

Solution

$T_1 = 6(1) + 2 = 8$ — Replace n with "1" for Term_1

$T_2 = 6(2) + 2 = 14$ — Replace n with "2" for Term_2

$T_3 = 6(3) + 2 = 20$

$T_4 = 6(4) + 2 = 26$ — $\therefore$ sequence is 8; 14; 20; 26; …

Example

Example 2

Given the sequence 3; 7; 11; 15 … determine the formula for T_n.

Solution

Solution

3 ; 7 ; 11 ; 15 …

+4 +4 (+4) → constant difference

We note that the difference between successive terms is constant. When this happens we know that our generating formula will be linear and of the form, $T_n = an + b$ where a is the constant difference.

So $T_n = 4n + b$.

Now sub $n = 1$: $T_1 = 4(1) + b = 3$

$b = -1$

so $Tn = 4n - 1$

Example

Example 3

Given the sequence 10; 2; –6; –14; … find T_n

Solution

Solution

10 ; 2 ; –6 ; –14

–8 –8 (–8) → constant difference

$\therefore T_n = -8n + b$

now $T_1 = -8(1) + b = 10$ (substitute $n = 1$)

$b = 18$

so $T_n = -8n + 18$

The general term of a sequence with a constant first difference "a" is

$T_n = an + b$

Sometimes however, you may get a sequence of numbers which does not follow this kind of pattern, which has a constant difference

Examine:

$$2\ ;\ 6\ ;\ 18\ ;\ 54\ ;\ \ldots$$

$+4 \quad +12 \quad +36$ ✗ wrong (no constant number)

$\times 3 \quad \times 3$

There is not a constant difference, but we do notice something else. Each term is the result of the previous one multiplied by 3. We say there is a constant ratio and we can predict that the next term would be 162.

Quadratic Formulae

This year in Grade 11, we specifically study sequences generated by a formula of the type: $T_n = an^2 + bn + c$. This is called a quadratic formula.

Example 4

Write down the first four terms of the sequence generated by

1. $T_n = n^2$

   $T_1 = (1)^2 = 1$

   $T_2 = (2)^2 = 4$

   $T_3 = (3)^2 = 9$

   $T_4 = (4)^2 = 16 \quad \therefore\ 1; 4; 9; 16; \ldots$

2. $T_n = n^2 + 3$

   $T_1 = (1)^2 + 3 = 4$

   $T_2 = (2)^2 + 3 = 7$

   $T_3 = (3)^2 + 3 = 12$

   $T_4 = (4)^2 + 3 = 19 \qquad \therefore\ 4; 7; 12; 19; \ldots$

Our task is to learn how to reverse the process. That is, if we are given the sequence we need to be able to determine the general term/formula.

Example 5

Given $2\ ;\ 5\ ;\ 10\ ;\ 17 \ldots$

$+3 \quad +5 \quad +7$ work out the first difference

$+2 \quad +2$ work out the second difference which is now constant

When this happens our generating formula (T_n) is the form $T_n = an^2 + bn + c$ and we say it is quadratic.

For this example the easiest way to get an answer for T_n is to realise that when

$T_n = n^2$ we get 1; 4; 9; 16; …

so if $T_n = n^2 + 1$ we get 2; 5; 10; 17; … (just add 1 on each)

likewise $T_n = n^2 + 2$ we get 3; 6; 11; 18; … (just add 2 on each)

also $T_n = 2n^2$ we get 2; 8; 18; 32; … (just double each)

However, most times, you will need a more comprehensive method to find T_n, than just seeing how it relates to n^2.

Let us look at any sequence with $T_n = an^2 + bn + c$

Now $T_1 = a(1)^2 + b(1) + c$ (sub $n = 1$)

$T_2 = a(2)^2 + b(2) + c$ (sub $n = 2$)

$T_3 = a(3)^2 + b(3) + c$ (sub $n = 3$)

$T_4 = a(4)^2 + b(4) + c$ (sub $n = 4$)

$$\begin{array}{ccccccc} T_1 & ; & T_2 & ; & T_3 & ; & T_4 \\ \therefore\ a+b+c & ; & 4a+2b+c & ; & 9a+3b+c & ; & 16a+4b+c;\ldots \end{array}$$

1st diff $3a + b$ $5a + b$ $7a + b$

2nd diff $2a$ $2a$

This explanation allows us to generalise that we can always equate our second difference to $2a$, and thereafter we can solve for a.

Example

Example 6

Find T_n if given the sequence 2; 7; 14; 23; 34; …

Solution

Solution

2 ; 7 ; 14 ; 23 ; 34 ; …

1st diff: +5 +7 +9 +11

2nd diff: +2 +2 +2

So set $2 = 2a$

$\therefore 1 = a^*$

So $T_n = an^2 + bn + c$ now becomes

$T_n = 1n^2 + bn + c$ (from *)

Now we need to solve for b and c, (two unknowns – so we need to set up two equations and solve simultaneously).

So $T_1 = 1(1)^2 + b(1) + c = 2$ and $T_2 = 1(2)^2 + b(2) + c = 7$

$1 + b + c = 2$ $4 + 2b + c = 7$

$c = 1 - b$ sub in for c ⟶ $2b + (1 - b) = 3$

$b = 2$

Using $b = 2$ ⟵ sub back $b = 2$

$c = 1 - (2)$

$c = -1$

so formula is $\boxed{T_n = n^2 + 2n - 1}$

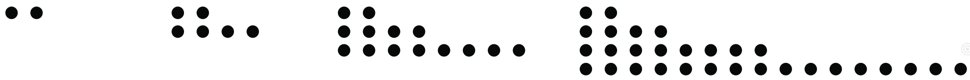
Always check your formula: $T_1 = (1)^2 + 2(1) - 1 = 2$ ✓

$T_2 = (2)^2 + 2(2) - 1 = 7$ ✓

Example 7

Dots are arranged in the following pattern:

Pattern 1 Pattern 2 Pattern 3 Pattern 4



1. Determine the number of dots in the nth pattern.

Solution

$$2 \; ; \; 6 \; ; \; 14 \; ; \; 30 \; ; \; 42 \; ; \; \ldots$$

1st diff: $+4 \quad +8 \quad +12 \quad +16.$

2nd diff: $+4 \quad +4 \quad +4$

So set $4 = 2a$

$2 = a$

$\therefore \; T_n = 2n^2 + bn + c$

Now $T_1 = 2 + b + c = 2$ and $T_2 = 8 + 2b + c = 6$

$c = -b$ $\quad 2b + c = -2$

sub in for c $\quad 2b + (-b) = -2$

$b = -2$

Using $b = -2$ sub back $b = -2$

$c = 2$

$\therefore \; \boxed{T_n = 2n^2 - 2n + 2}$

Check: $T_1 = 2 - 2 + 2 = 2$ ✓

$T_2 = 8 - 4 + 2 = 6$ ✓

On the last examples we have used a very successful method. However, if you are willing to learn a rather long formula given by

$T_n = T_1 + (n-1)s + \frac{(n-1)(n-2)}{2} f$ where s is the first term of the first difference row, and f is the value of the second difference then this is a worthwhile alternative.

Let us use the above example to demonstrate

$T_n = an^2 + bn + c$

$T_n = T_1 + (n-1)s + \frac{(n-1)(n-2)}{2} \cdot f$

$T_n = 2 + (n-1)4 + \frac{(n-1)(n-2)}{2} \cdot 4 \quad \left(\text{cancel } \frac{4}{2} = 2\right)$

$T_n = 2 + 4n - 4 + 2\,(n-1)(n-2)$

$T_n = 2 + 4n - 4 + 2n^2 - 6n + 4$

$\boxed{T_n = 2n^2 - 2n + 2}$

2. Determine how many dots the 30th pattern will have

Solution

The 30th pattern will have $2(30)^2 - 2(30) + 2 = 1\,742$ dots.

You are now ready for the last worked examples.

Example 3

Find T_n of the sequence 18; 13; 9; 6; 4; …

Solution

$$18 \;;\; 13 \;;\; 9 \;;\; 6 \;;\; 4 \;;\; \ldots$$

1st diff $\quad -5 \quad -4 \quad -3 \quad -2$ $\quad\rightarrow\quad 2a = 1$

2nd diff $\quad +1 \quad +1 \quad +1$ $\quad\quad a = \frac{1}{2}$

So $T_n = \frac{1}{2}n^2 + bn + c$

Method 1

$T_1 = \frac{1}{2} + b + c = 18$

$b = 17\frac{1}{2} - c$

$T_2 = 2 + 2b + c = 13$

$c = 11 - 2b$

so $b = 17\frac{1}{2} - (11 - 2b)$

$b = \frac{-13}{2}$

and using $b = \frac{-13}{2}$

$c = 24$

so $\boxed{T_n = \frac{1}{2}n^2 - \frac{13}{2}n + 24}$

Method 2

$T_n = 18 + (n - 1)(-5) + \frac{(n-1)(n-2)}{2} \times$

$= 18 - 5n + 5 + \frac{n^2 - 3n + 2}{2}$

$= \frac{36 - 10n + 10n + 10 + n^2 - 3n + 2}{2}$

$= \frac{n^2 - 13n + 48}{2}$

$\boxed{T_n = \frac{1}{2}n^2 - \frac{13}{2}n + 24}$

Perhaps easier if you know the formula

Check $T_1 = \frac{1}{2} - \frac{13}{2} + 24 = 18$ ✓

$T_2 = \frac{1}{2}(4) - \frac{13}{2}(2) + 24 = 13$ ✓

As seen a little earlier finding T_n for a quadratic sequence is often related to patterns.

Example 4

Given $\quad 1 + 2 = 3$

$4 + 5 + 6 = 7 + 8$

$9 + 10 + 11 + 12 = 13 + 14 + 15$

$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$

1. Find the first term of the fifteenth row.
2. Find the first term that occurs immediately before the equals sign "=" in the n^{th} row.
3. Find the term just before the "=" sign in the 20^{th} row.

Solution

1. If we look at the first terms in all the rows they form their own sequence

   1; 4; 9; 16; …

   so $T_1 = (1)^2$

   $T_2 = (2)^2$

$T_3 = (3)^2$

so first term in fifteenth row would be $T_{15} = (15)^2 = 225$

2. If we look at the terms before the equal sign of each row, they also form a pattern.

$$2 \; ; \; 6 \; ; \; 12 \; ; \; 20 \; ; \; \ldots$$

$$+4 \quad +6 \quad +8$$

$$+2 \quad +2$$

So $T_n = n^2 + bn + c$ (since $a = 1$)

$T_1 = 1 + b + c = 2$ — sub $b = 1 - c$ → $T_2 = 4 + 2b + c = 6$

$= 4 + 2(1 - c) + c = 6$

$c = 0$ — sub back $c = 0$ →

get $b = 1$

so $T_n = n^2 + n$

Check $T_1 = 1^2 + 1$ ✓

$T_2 = 2^2 + 2$ ✓

3. $T_{20} = (20)^2 + (20)$

$= 420$

Can you imagine if we tried to do this without the aid of a formula? It would take ages to write out all the numbers till we hit the twentieth row!

Not all sequences of numbers can be generated by a linear general term

($T_n = an + b$) or a quadratic general term ($T_n = an^2 + bn + c$)

Examine the following sequences:

1. 1; 8; 27; 64; 125; … We recognise these numbers as the perfect cubes

$(1)^3; (2)^3; (3)^3; (4)^3; (5)^3$

so $T_n = n^3$

2. 6; 12; 24; 48; … We see this as

$2 \times 3;\ 4 \times 3; 8 \times 3; 16 \times 3; \ldots$

$2^1 \times 3; 2^2 \times 3; 2^3 \times 3; 2^4 \times 3; \ldots$

so $T_n = 2^n.3$

3. $20; -10; 5; -2\frac{1}{2}; \ldots$ We see this as

$20; 20 \times -\frac{1}{2}; 20 \times -\frac{1}{2} \times -\frac{1}{2}; 20 \times -\frac{1}{2} \times -\frac{1}{2} \times -\frac{1}{2}$

$20; 20 \times \left(-\frac{1}{2}\right); 20 \times \left(-\frac{1}{2}\right)^2; 20 \times \left(-\frac{1}{2}\right)^3$

so $T_n = 20\left(-\frac{1}{2}\right)^{n-1}$

Activity 1

1. Consider the following sequences and in each case:

a. Write down the next 3 terms.

b. State whether the general term of the sequence will be liner, quadratic or neither.

1.1 5; 9; 13; 17; …

1.6 3; –9; 27; –81; …

1.2 45; 39; 33; 27; …

1.7 2; –1; –6; –13; …

1.3 0; 2; 6; 12; …

1.8 $3; 10\frac{1}{2}; 20; 31\frac{1}{2}; \ldots$

1.4 $2; 3\frac{1}{2}; 5; 6\frac{1}{2}; 8; \ldots$

1.9 $4 + 3x; 5 + 5x; 6 + 7x$

1.5 1; 2; 4; 8; 16; …

1.10 $4 + 3x; 5 + 5x; 7 + 9x; 10 + 15x$

2. In each case the general term (T_n) is given. Write down the first three terms for each.

2.1 $T_n = 4n - 3$

2.5 $T_n = n^2 - n + 4$

2.2 $T_n = 2n^2 - 1$

2.6 $T_n = -\frac{1}{2}n^2 + n$

2.3 $T_n = 2^{-n}$

2.7 $T_n = n^3 + 2$

2.4 $T_n = 5.3^{n-1}$

2.8 $T_n = \frac{n+1}{n^2}$

Activity

Activity 2 (A little more challenging!)

1. Extend the following sequences by three terms:

1.1 5; –1; –7; …

1.4 1; 1; 2; 3; 5; …

1.2 8; –4; 2; …

1.5 $\frac{2}{4}; \frac{4}{7}; \frac{6}{10}; \ldots$

1.3 3; 8; 15; …

1.6 $\frac{2}{3}; \frac{4}{5}; \frac{6}{9}; \frac{8}{15}; \ldots$

2. Find

2.1 a formula for the general term T_n and

the number of terms in the given sequence

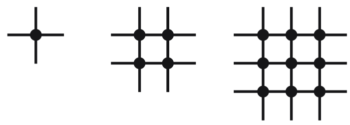
2.1.1 6; 10; 14; 18; …; 222

2.1.2 5; 10; 17; 26; … 10202

2.1.3 $\frac{2}{7}; \frac{5}{9}; \frac{8}{13}; \frac{11}{19}; \ldots; \frac{89}{877}$

2.1.4 6; 4; 0; –6; …; –84

3. Johnny puts toothpicks into jelly-tots as shown.



3.1 Copy and complete the table below.

Pattern number	Number of Jelly Tots	Number of toothpicks
1	1	
2	4	
3	9	
4		
5		
n		

3.2 Hence find the number of toothpicks in the n^{th} pattern

4. Thato and Thandi were doing a Maths assignment and were asked to examine the sequence 3; 6; 12; …

Thato says that the next two terms are 24 and 48. Thandi disagrees and says that the next two terms are 21 and 33.

4.1 Explain why they could both be correct.

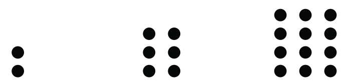
4.2 Find T_n, the general term in both cases.

4.3 Calculate whether 1 005 could possibly be a term in Thato's sequence. Explain.

4.4 Which term in Thandi's sequence is equal to 111.

5. Chris and Richard look at the following pattern:

Pattern 1 Pattern 2 Pattern 3



Chris says that he can work out the general formula for the number of dots by taking the number of columns in that pattern and multiplying that by the number of rows.

Richard says that if he increases the pattern number by one, squares that and then subtracts the pattern number increased by one he will get the general formula. Check who is correct.

6. Examine the following sequences and answer the questions that follow:

1; 2; 4; 5; 7; 8; 10; 11; 13; … 118

1 2; 4; 5; 7; 10; 10; 17; 13; …; 101

6.1.1 Give the next three numbers in the sequence after the 13.

6.1.2 Describe the pattern recognisable in words.

6.1.3 Calculate the number of terms in each sequence.

Solutions to Activities

Activity 1

1.1a 21; 25; 29

1.1b Linear

1.2a 21; 15; 9

1.2b Linear

1.3a 20; 30; 42

1.3b Quadratic

1.4a $9\frac{1}{2}; 11; 12\frac{1}{2}$

1.4b Linear

1.5a 32; 64; 128

1.5b Neither

1.6a 243; –729; 2 187

1.6b Neither

1.7a –22; –33; –46

1.7b Quadratic

1.8a $45; 60\frac{1}{2}; 78$

1.8b Quadratic

1.9a $7 + 9x; 8 + 11x; 9 + 13x$

1.9b Linear

1.10a $14 + 23x; 19 + 33x; 25 + 45x$

1.10b Quadratic

2.1 1; 5; 9

2.2 1; 7; 17

2.3 $\frac{1}{2}; \frac{1}{4}; \frac{1}{8}$

2.4 5; 15; 45

2.5 4; 6; 10

2.6 $\frac{1}{2}; 0; -1\frac{1}{2}$

2.7 3; 10; 29

2.8 $2; \frac{3}{4}; \frac{4}{9}$

Activity 2

1.1 5; –1; –7; –13; –19; –25

1.2 $8; -4; 2; -1; \frac{1}{2}; -\frac{1}{4}$

1.3 3; 8; 15; 24; 35; 48

1.4 1; 1; 2; 3; 5; 8; 13; 21;

1.5 $\frac{2}{4}; \frac{4}{7}; \frac{6}{10}; \frac{8}{13}; \frac{10}{16}; \frac{12}{19}$

1.6 $\frac{2}{3}; \frac{4}{5}; \frac{6}{9}; \frac{8}{15}; \frac{10}{23}; \frac{12}{33}; \frac{14}{45}$

2.1.1 $\underbrace{6 \; ; \; 10}_{4} \; ; \underbrace{\; 14}_{4} \; ; \underbrace{\; 18}_{4} \; ; \ldots \; 222$

$T_n = 4n + 2$

$222 = 4n + 2$

$220 = 4n$

$55 = n$

$\therefore T_{55} = 222$

$\therefore$ 55 terms in sequence

2.1.2 $5 \; ; \; 10 \; ; \; 17 \; ; \; 26 \; ; \ldots ; 10202$

first differences: 5, 7, 9

second differences: 2, 2

$T_n = n^2 + bn + c$ $\qquad 2a = 2$

$\therefore a = 1$

$T_1 = 1 + b + c = 5$ $\qquad T_2 = 4 + 2b + c = 10$

$b + c = 4$ $\qquad 2b + 4 - b = 10 - 4$

$c = 4 - b$ $\qquad b = 2$

$c = 2$

$\therefore T_n = n^2 + 2n + 2$

$n^2 + 2n + 2 = 10202$

$n^2 + 2n - 10200 = 0$

$(n - 100(n + 102) = 0$

$n = 100; n = -102$

$\therefore$ 100 terms in sequence

2.1.3 $\frac{2}{7}; \frac{5}{9}; \frac{8}{13}; \frac{11}{19}; \ldots \frac{89}{877}$

Numerator $T_n = 3n - 1$ $\qquad \therefore T_n = \frac{3n-1}{n^2-n+7}$

Denominator $T_n = n^2 - n + 7$

Numerator $3n - 1 = 89$

$3n = 90$

$n = 30$

or an even harder method is to put $n^2 - n + 7 = 877$

$n^2 - n - 870 = 0$

$(n + 29)(n - 30) = 0$

$n = -29$ N/A; $n = 30$

so either way we go about it, we get that there must be 30 terms in this sequence.

2.1.4 Look at $6 \; ; \; 4 \; ; \; 0 \; ; -6; \ldots; -84$

$-2 \quad -4 \quad -6$ 1st difference

$-2 \quad -2$ 2nd difference is now constant.

So as usual take $2a = -2$

$a = -1$

So $T_n = -n^2 + bn + c$ $\qquad T_2 = -4 + 2b + c = 4$

$T_1 = -1 + b + c = 6$ $\qquad 2b + c = 8$

sub $c = 7 - b$ $\qquad 2b + (7 - b) = 8$

$b = 1$

using $b = 1$

$c = 6$

so $T_n = -n^2 + n + 6$

$-n^2 + n + 6 = -84$

$n^2 - n - 90 = 0$

$(n - 10)(n + 9) = 0$

$n = 10$ OR $n = -9$ N/A

So we see that T_{10} would be equal to –84.

3.1

Pattern number	Number of Jelly Tots	Number of toothpicks
1	1	4
2	4	12
3	9	24
4	16	40
5	25	60
n	n^2	

First differences: 8; 12; 16; 20. Second differences: 4; 4; 4.

$2a = 4$

$a = 2$

3.2 So for number of toothpicks $T_n = 2n^2 + bn + c$

So $T_1 = 2 + b + c = 4 T_2 = 8 + 2b + c = 12$

$b = 2 - c$ $\quad 8 + 2(2 - c) = 12$

$8 + 4 - 2c = 12$

$b = 2 - 0$ $\quad c = 0$

$b = 2$ using $c = 0$

So last line in n^{th} row will be $T_n = 2n^2 + 2n$.

This is the number of toothpicks in the n^{th} pattern.

4.1 Thato 3; 6; 12; 24; 48 …

He sees this as a sequence where each term is doubled.

Thandi 3; 6; 12; 21; 33 …

She sees this as a sequence with a second order difference of 3.

4.2 Thato $T_1 = 3$; $T_2 = 3 \times 2$; $T_3 = 3 \times 2^2$; $T_n = 3 \times 2^{n-1}$

Thandi $T_n = \frac{3}{2}n^2 + bn + c$

$T_1 = \frac{3}{2} + b + c = 3$ $\quad T_2 = \frac{3}{2}(4) + 2b + c = 6$

$b = 1\frac{1}{2} - c$ $\quad 6 + 2\left(1\frac{1}{2} - c\right) + c = 6$

$3 - 2c + c = 0$

$c = 3$

using $c = 3$

$b = -1\frac{1}{2}$

$\therefore T_n = \frac{3}{2}n^2 - \frac{3}{2}n + 3$

4.3 For Thato 3; 6; 12; 24; 48; …

All terms except T_1 are even, therefore 1005 cannot be part of sequence.

4.4 $\frac{3}{2}n^2 - \frac{3}{2}n + 3 = 111$

$3n^2 - 3n + 6 - 222 = 0$

$3n^2 - 3n - 216 = 0$

$n^2 - n - 72 = 0$

$(n - 9)(n + 8) = 0$

$n = 9; n = -8$ N/A

$\therefore$ 9th term is equal to 111

5. Chris says $T_n = n(n + 1) = n^2 + n$

Richard says $T_n = (n + 1)^2 - (n + 1) = n^2 + 2n + 1 - n - 1$

$= n^2 + n$

They are both the correct formula for the sequence 2; 6; 12; …

6.1.1. 1; 2; 4; 5; 7; 8; 10; 11; 13; 14; 16; 17; … 118

1; 2; 4; 5; 7; 10; 10; 17; 13; 26; 16; 37; … 101

6.1.2. The sequence is actually two linear sequences that are merged

1; 4; 7; 10; 13; 16; … merges with 2; 5; 8; 11; 14; 17; …

The sequence is actually a linear sequence merged with a quadratic one

1; 4; 7; 10; 13; 16 … merges with 2; 5; 10; 17; 26; 37; …

6.1.3. 1; 4; 7; 10 … $T_n = 3n - 2$ or $T_n = 3n - 1$ (for 2; 5; 8; 11 …)

$118 = 3n - 2$ $\quad$ $118 = 3n - 1$

$120 = 3n$ $\quad$ $119 = 3n$

$40 = n$ $\quad$ $\frac{119}{3} \neq n$ $\therefore$ not a term in this sequence

$\therefore$ 118 is the 40th term of this sequence – there are 39 terms of the other sequence so 118 is the 79th term.

1; 4; 7; 10; 13; 16; … $T_n = 3n - 2$ $\quad$ 2; 5; 10; 17; 26; 37; …

$111 = 3n - 2$ $\quad$ $T_n = n^2 + 1$

$113 = 3n$ $\quad$ $n^2 + 1 = 101$

$\frac{113}{3} \neq n$ $\quad$ $n^2 = 100$

$n = 10$

$\therefore$ 101 is the 10th term of this sequence – there are 10 terms of the other sequence so 101 is the 20th term.