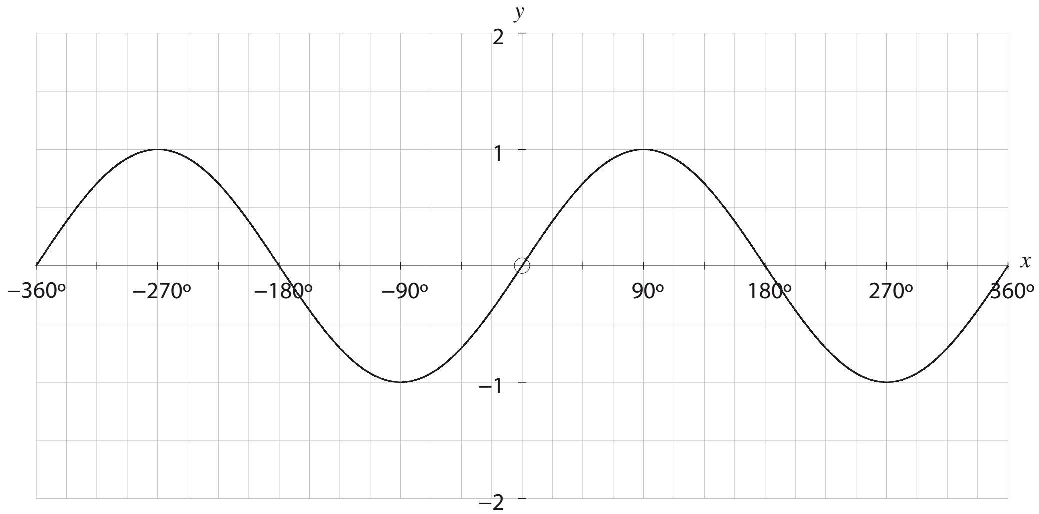
FUNCTIONS

Trigonometric graphs

In grade 10, you studied the graphs of $y = \sin x$, $y = \cos x$ AND $y = \tan x$

Let us recall these.

The graph of $y = \sin x$

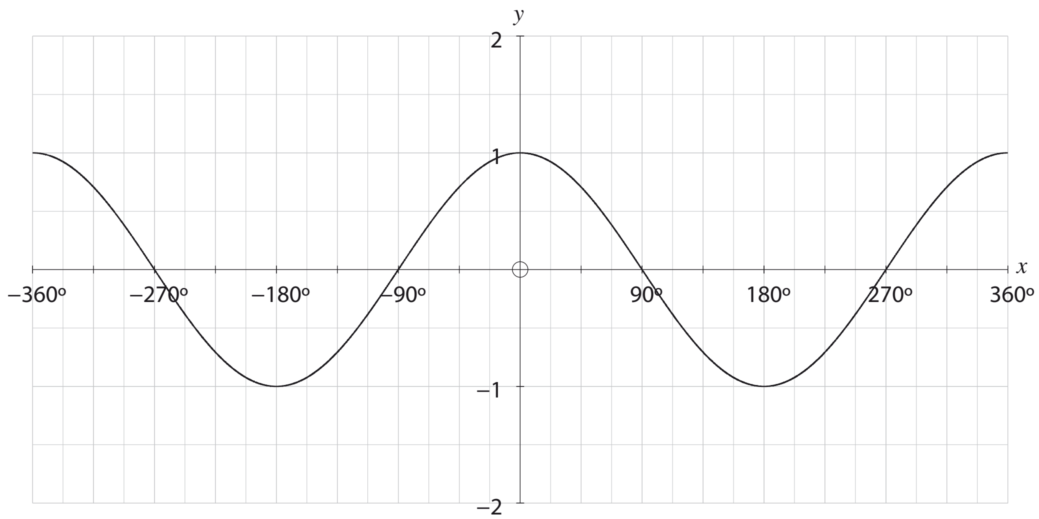


The graph of $y = \sin x$ has a period of 360°, i.e. it repeats itself after every 360°.

The graph has amplitude of 1, i.e all y values lie between 1 and –1.

The graph goes through the origin.

The graph of $y = \cos x$



The graph of $y = \cos x$ has a period of 360°, i.e. it repeats itself after every 360°.

The graph has amplitude of 1, i.e. all y values lie between 1 and –1.

The graph cuts the y-axis at $y = 1$.

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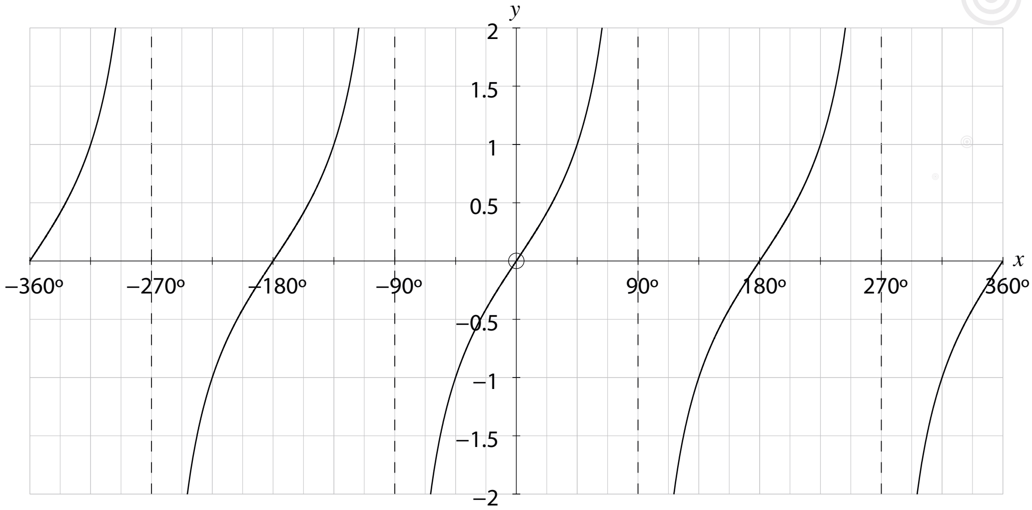
The graph of $y = \tan x$ is very different from the graphs of $y = \sin x$ and $y = \cos x$.



The graph repeats itself every 180°, i.e. the graph of $y = \tan x$ has a period of 180°.

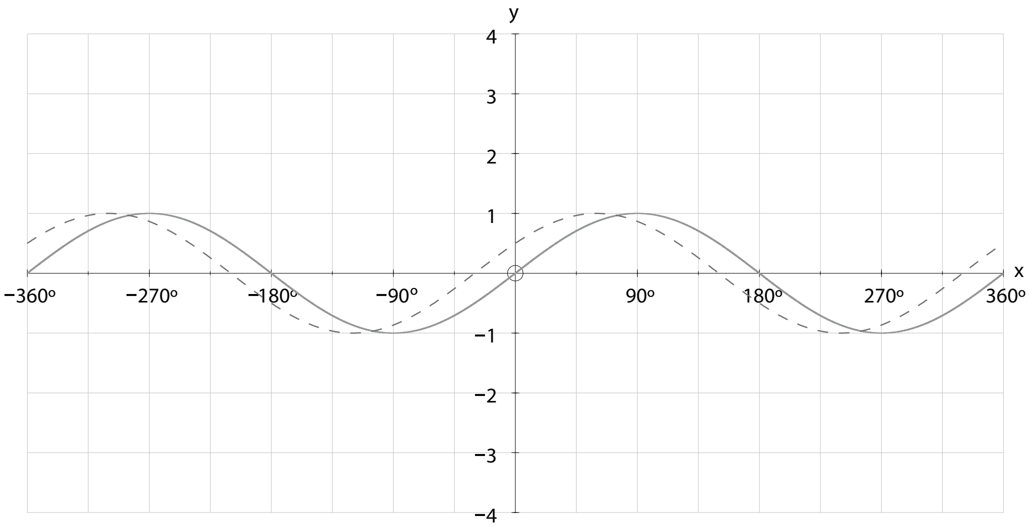
The graph also has asymptotes at $x = -270°$ x; $= -90°$; $x = 90°$ and $x = 270°$.

The graph goes through the origin.

In Grade 11 we look at the effect of adding p to the x, i.e we look at the graphs of $y = \sin(x + p)$; $y = \cos(x + p)$ and $y = \tan(x + p)$.

Study the pairs of graphs below and see if you can describe the effect.

Let us first look at positive values of p.



The graph of $y = \sin(x + 30)$ is easily sketched by plotting points obtained from a table using the CASIO fx-82ES PLUS.

Step 1: Press Mode. Choose option 3: TABLE.

Enter $\sin(x + 30)$, so that your screen looks as follows $f(x) = \sin(x + 30)$

Step 2: The screen will prompt a starting value by showing

Start?

Enter –360 and then the equal sign.

Step 3: The screen will prompt an ending value by showing

End?

Enter 360 and then the equal sign

Step 4: Now the screen displays

Step?

This is the value that represents the interval between points.

Type 30 and then press the equal sign.

The calculator now displays the following table with points

–360	0,5
–330	0,866
–300	1
–270	0,866
–240	0,5
–210	0
–180	–0,5
–150	–0,866
–120	–1
–90	–0,866
–60	–0,5
–30	0
0	0,5
30	0,866
60	1
90	0,866
120	0,5
150	0
180	–0,5
210	–0,866
240	–1
270	–0,866
300	–0,5
330	0
360	0,5

All that is left is for you to plot these points and join them with a sine curve.

You can use the calculator to get a table of points for $y = \cos(x + 45)$. This time use 45 for your step.

The graph of $y = \cos(x + 45)$

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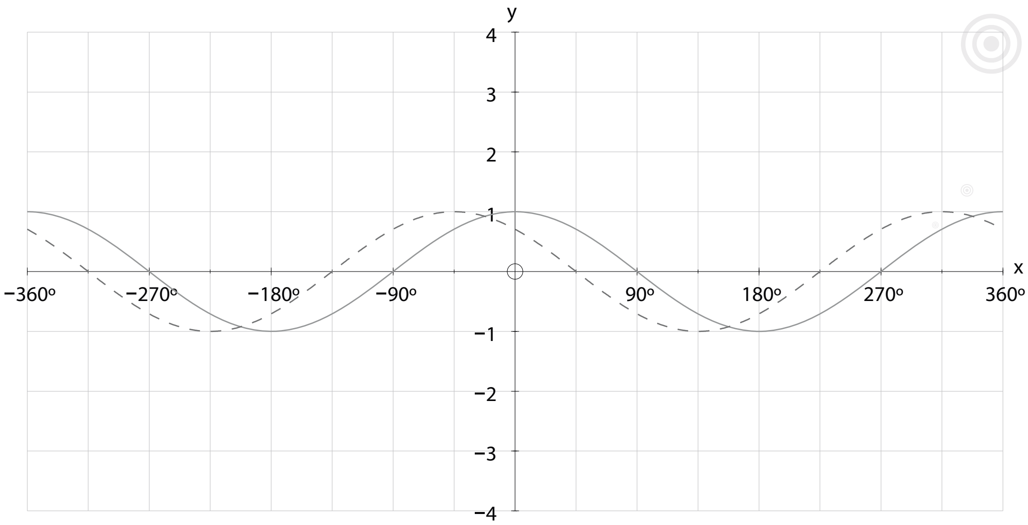
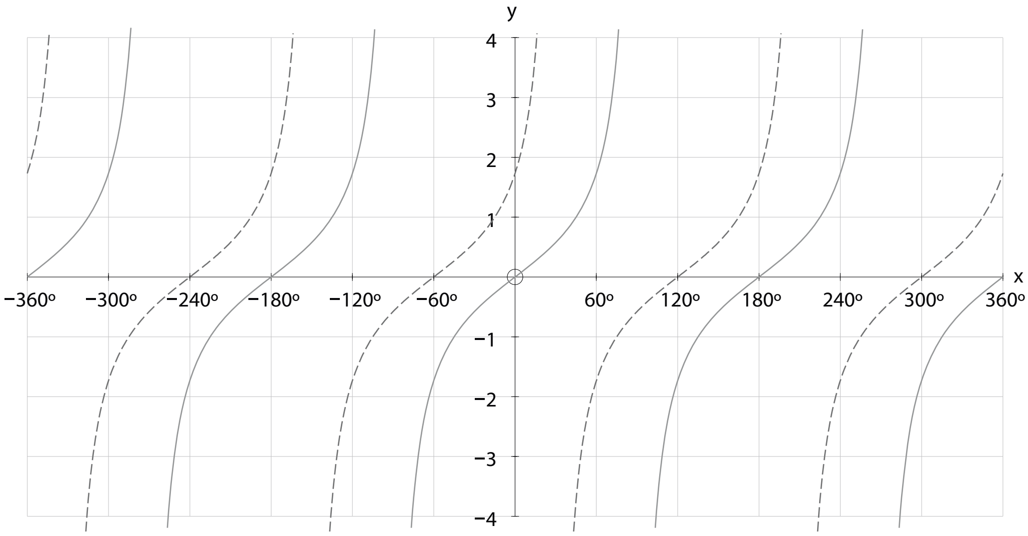


The graph of $y = \tan(x + 60°)$



If you are using the calculator to get a table of points to plot, use 30 as your step value and not 60.

Notice above that the asymptotes for $y = \tan x$ are $x = -90°$; $x = -270°$; $x = 90°$ and $x = 270°$.

Yet, the asymptotes for $y = \tan(x + 60°)$ are $x = -150°$; $x = -330°$; $x = 30°$ and $x = 210°$.

Are you able to describe the effect of the p in the equations $y = \sin(x + p)$; $y = \cos(x + p)$ and $y = \tan(x + p)$.

If you look carefully you will see that in each case the shape of the original graph has remained intact although it has shifted horizontally to the LEFT!! Observe how the positions of the asymptotes above changed.

What if the values of p are negative?

Again study the pairs of graphs given and see if you can describe the effect of adding a negative p value to x.

The graph $y = \tan(x - 60°)$

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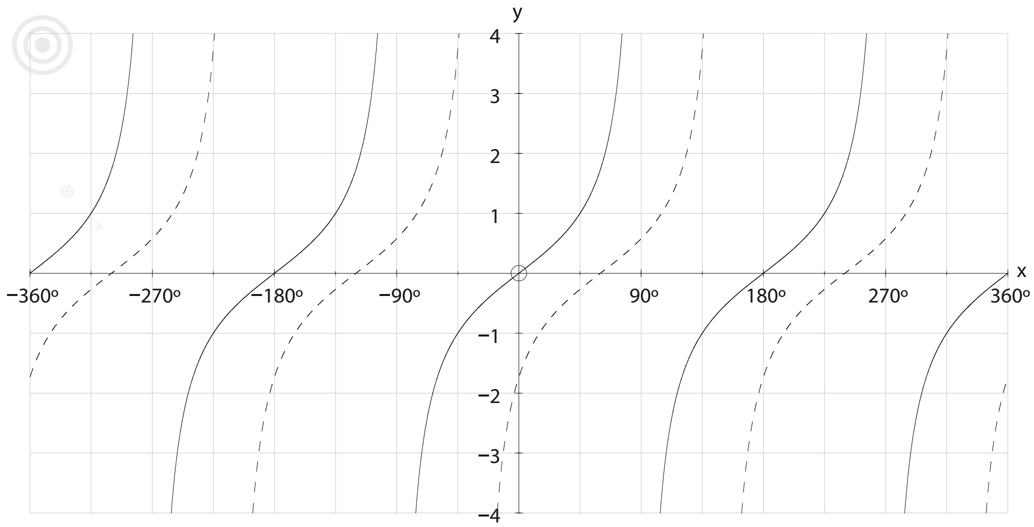


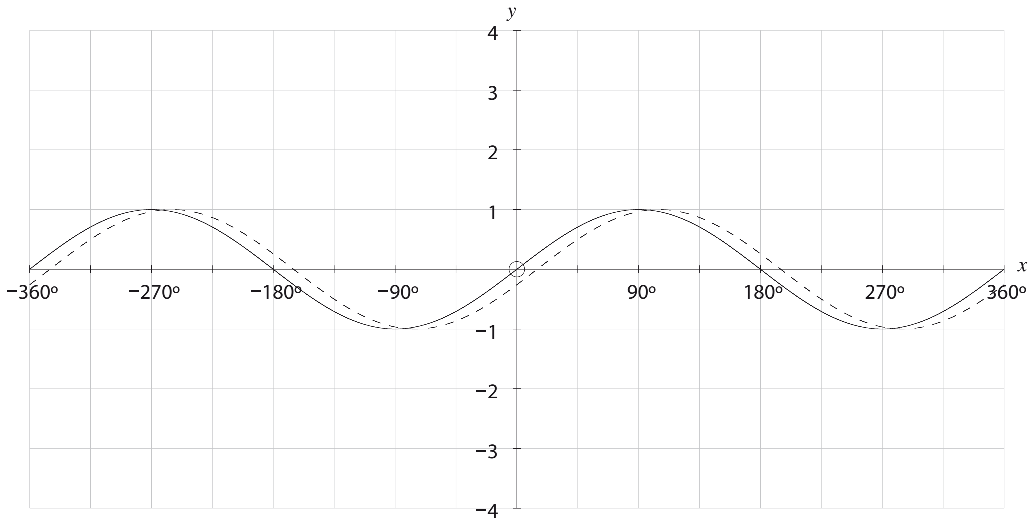
Look at the positions of the asymptotes of $y = \tan(x - 60°)$.

$x = -30°$; $x = -210°$; $x = 150°$; $x = 330°$.

Instead of the graph shifting to the left it has now shifted to the RIGHT!!!

The period of $y = \tan(x - 60°)$ remains 180°

The graph of $y = \sin(x - 15°)$



Again the graph of $y = \sin x$ has shifted to the right by 15°.

Note that the period remains unchanged, i.e. it is 360°

Also the amplitude remains unchanged, i.e. it is 1.

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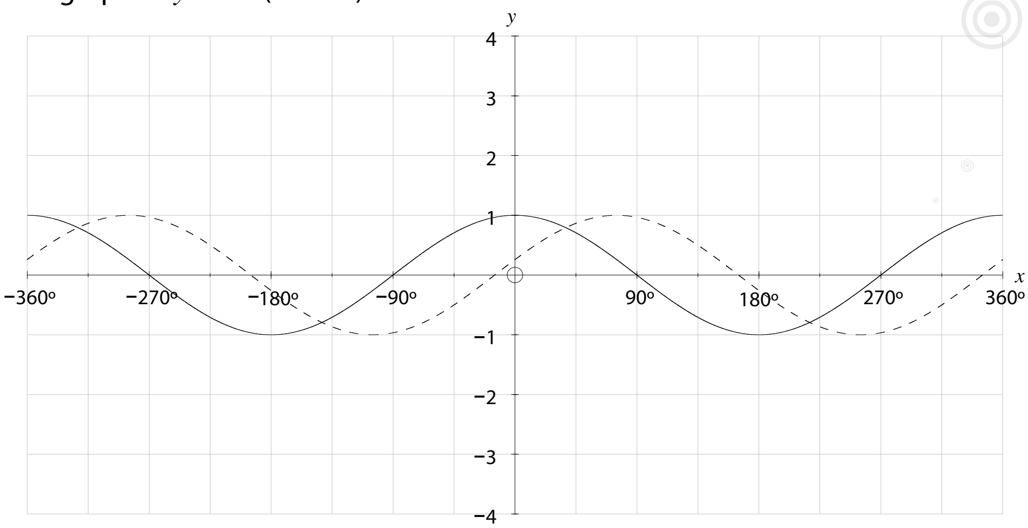
The graph of $y = \cos(x - 75°)$



By studying the graphs above we see that the graph of $y = \cos x$ has shifted horizontally to the right by 75°. The amplitude of the graph is still 1 and the period is still 360°.

In general,

if $p > 0$ then the graph shifts horizontally to the LEFT

and if $p < 0$ then the graph shifts horizontally to the RIGHT

Activity 1

Activity

1.1 On the same set of axes sketch the graphs of $f(x) = \sin(x - 30°)$ and $g(x) = \cos(x + 60°)$ for $x \in [-360°; 360°]$

1.2 What do you notice?

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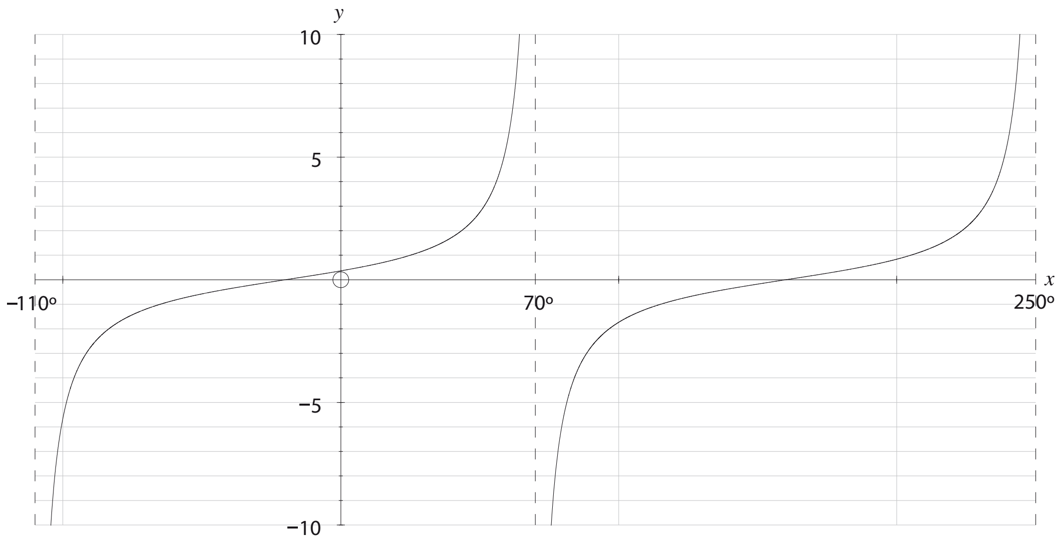
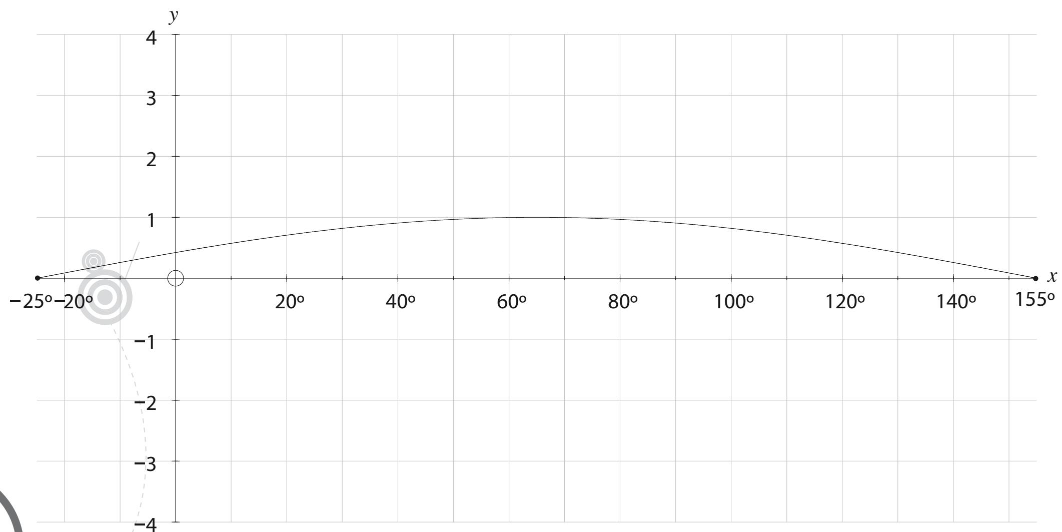
1.3 Can you explain your observation in 1,2?

2 In each case, give the value of p

2.1 $y = \tan(x + p)$



2.2 $y = \sin(x + p)$



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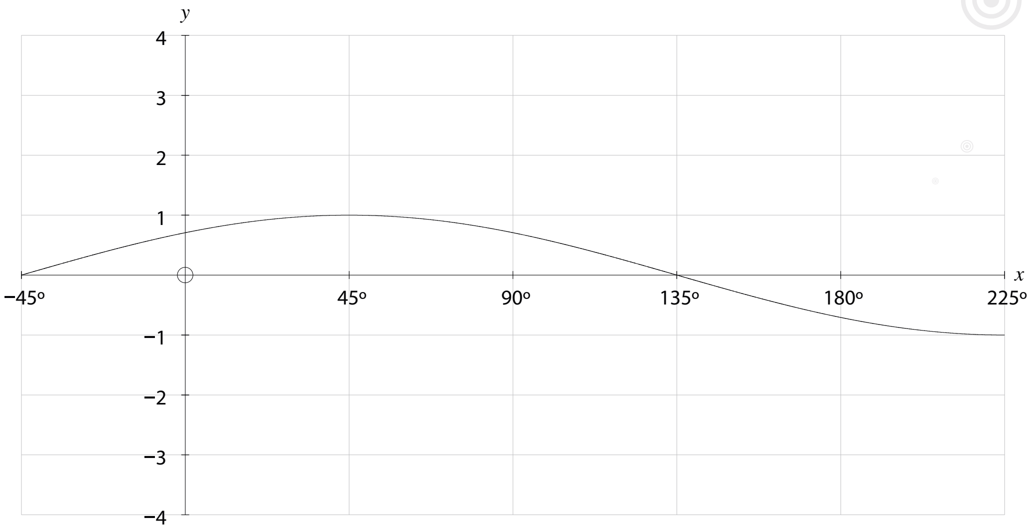
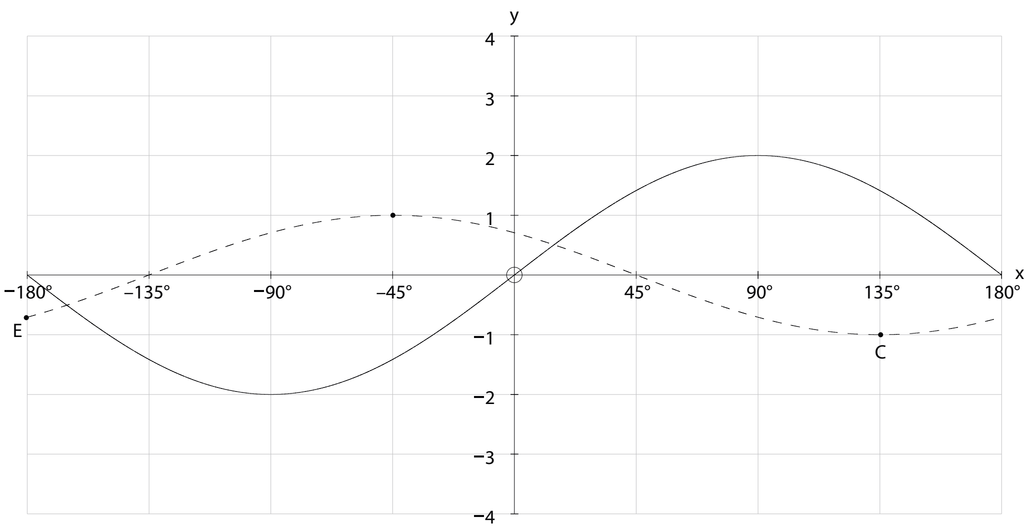
2.3 $y = \cos(x + p)$



3. The graphs of $f(x) = b\sin x$ and $g(x) = \cos(x + k)$ are sketched for $x \in [-180°; 180°]$.



3.1 Write down the values of b and k

3.2 What is the period of g?

3.3 Write down the coordinates of C if C is a turning point on g

3.4 Write down the coordinates of E

3.5 Determine the equation of f if the y-axis is moved 20° to the left.

The graphs of $y = \sin kx$; $y = \cos kx$; $y = \tan kx$

Up to now, the trig graphs you have sketched have all had a period of 360° in the case of sine and cosine and a period of 180° in the case of a tangent graph.

If the coefficient of x is changed from 1, then the period of the graph will change from 360° (sine and cosine) and 180° (tangent)

> In general,
>
> The period of $y = \sin(kx)$ is $\left(\frac{360°}{k}\right)$.
>
> The period of $y = \cos(kx)$ is $\left(\frac{360°}{k}\right)$
>
> The period of $y = \tan(kx)$ is $\left(\frac{180°}{k}\right)$
>
> Graphs (contrary to intuition) have reduced period if $k > 1$ and the period increases if $0 < k < 1$

Activity

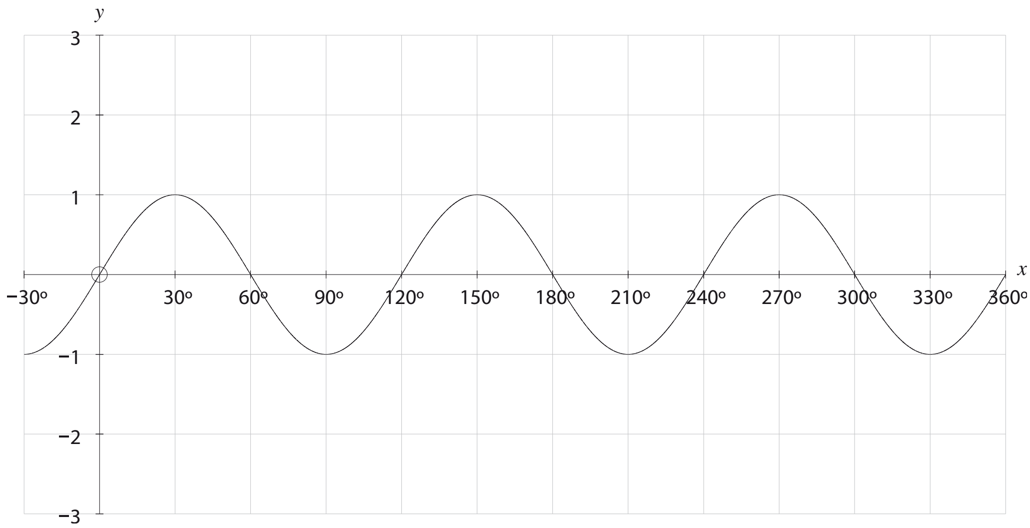
Activity 2

Without drawing the graph, write down the period of each of the following graphs

1. $y = \sin(3x)$ 2. $y = \cos(2x)$ 3. $y = \tan\left(\frac{1}{2}x\right)$

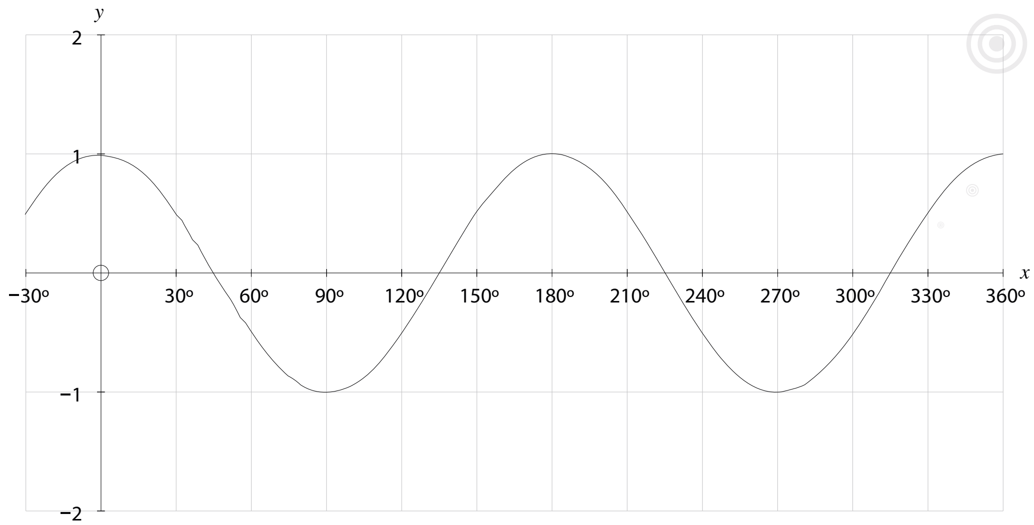
Now let us look at the graphs of the equations above.

Firstly, the graph of $y = \sin(3x)$.



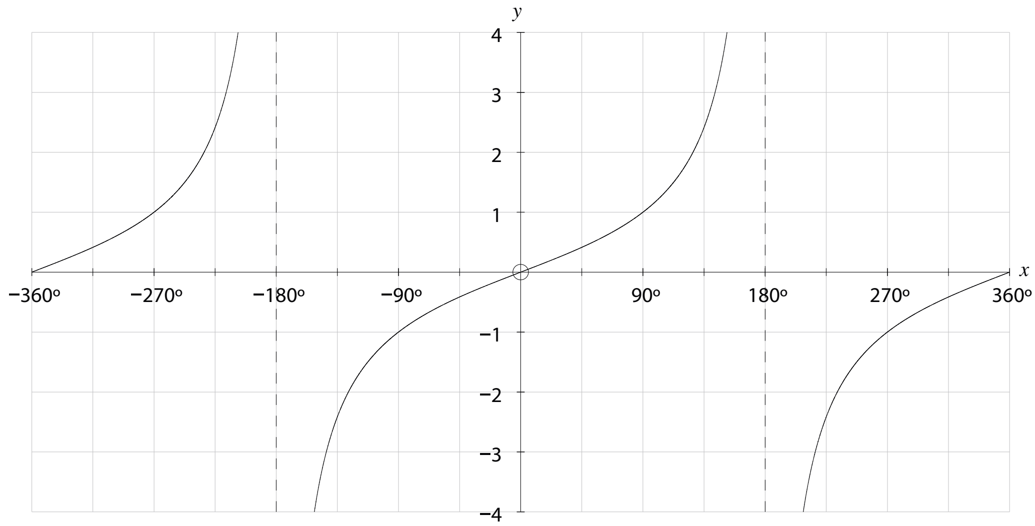
We see that the wave repeats itself after every 120°. There are 3 complete waves from 0° to 360°. For this reason the period is one third of the period of $y = \sin x$.

Secondly, the graph of $y = \cos(2x)$



We see that the wave repeats itself after every 180°. There are 2 complete waves from 0° to 360°. For this reason the period is one half of the period of $y = \cos x$.

Finally, let us look at $y = \tan\left(\frac{1}{2}x\right)$



This graph now repeats itself after every 360°. The original $y = \tan x$ graph repeated itself after every 180°.

The positions of the asymptotes have changed and their equations are now $x = 180°$ and $x = -180°$.

Drawing the graphs of $y = \sin kx$; $y = \cos kx$; $y = \tan kx$

Again, these are easily done by using the CASIO *ƒx*-82 ES PLUS.

Suppose you are asked to sketch the graph of $y = \cos(4x)$ for $x \in [-90°; 90°]$.

Step 1: Mode 3: TABLE and Enter the cos $(4x)$.

Step 2: Start –90 and press equals

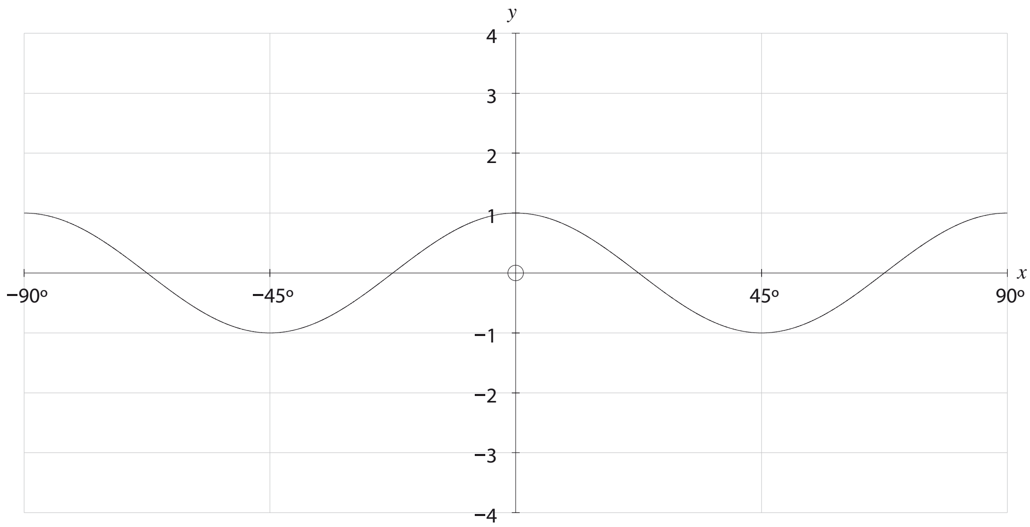
Step 3: End 90 and press equals

Step 4: This is the part that changes now as the period is no longer 360°. To get the period of $y = \cos(4x)$ we divide 360° by 4 and get 90°. In the previous examples we used 30 as our step. Dividing 90 by 4 gives us 22,5. This is the value you should use for the step.

Step 5: Plot the points from the table, which would look as follows:

–90	1
–67,5	0
–45	–1
–22,5	0
0	1
22,5	0
45	–1
67,5	0
90	1

Plotting these points we get the following graph.



Activity

Activity 3

1.1 On the same set of axes sketch the graphs of $f(x) = \sin 2x$ and $g(x) = 2\sin x$ for $x \in [0°; 180°]$

1.2 Write down $f \cap g$ for $x \in [0°; 180°]$

1.3 Use your graphs to solve the following inequalities if $x \in [0°; 180°]$

1.3.1 $(\sin 2x)(2\sin x) < 0$

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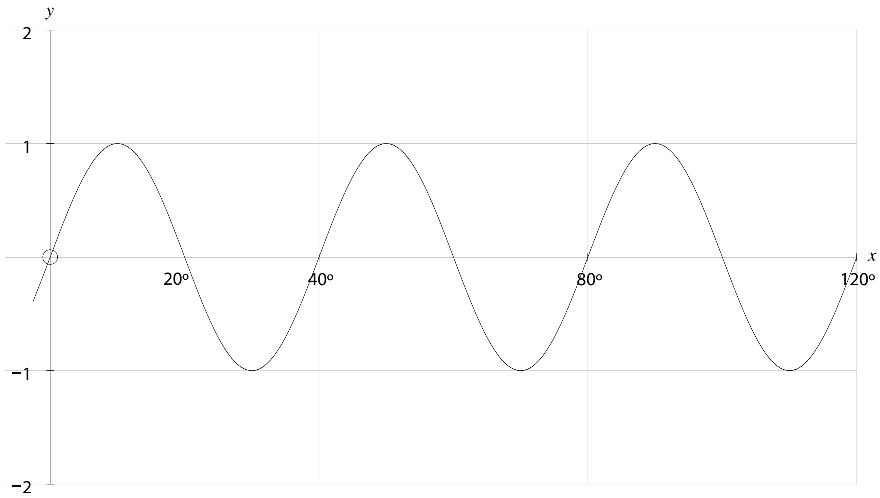
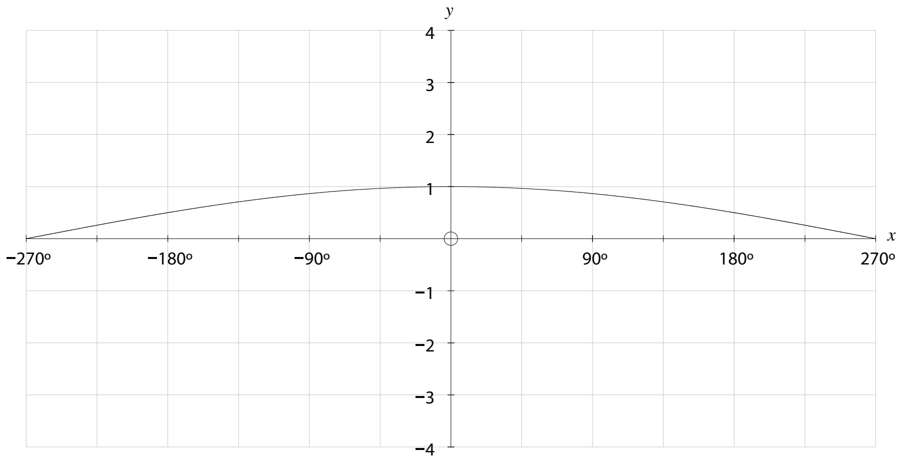
1.3.2 $\sin 2x > 2\sin x$

2. In each case, give the value of k and the period of the graph.

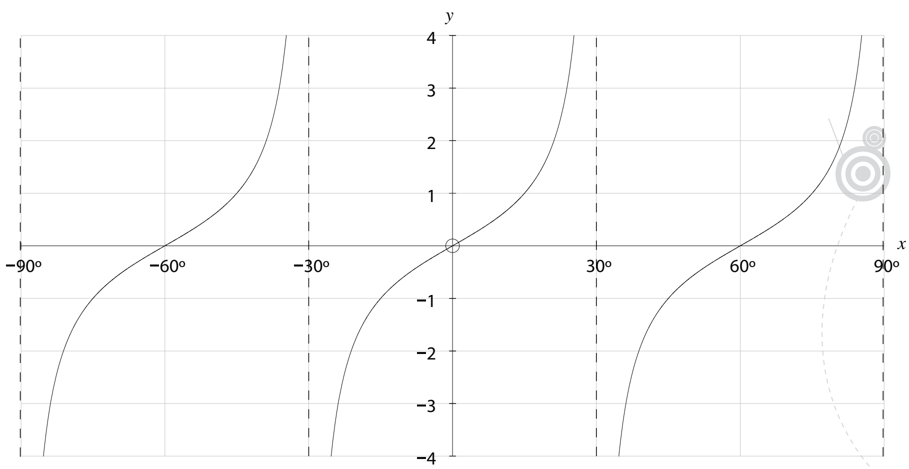
2.1 $y = \sin(kx)$



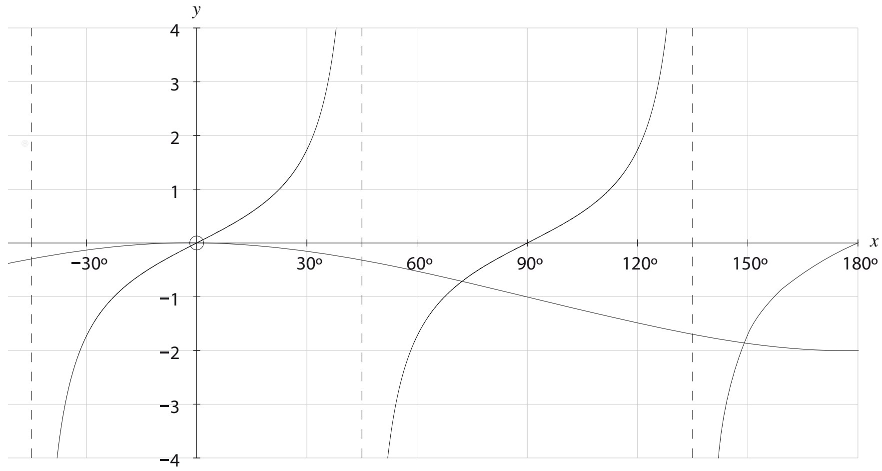
2.2 $y = \cos(kx)$



2.3 $y = \tan(kx)$



3. The graphs of $f(x) = a\cos x + b$ and $g(x) = \tan(kx)$ are sketched below for $x \in [-45°; 180°]$.



3.1 Determine the values of a, b and k

3.2 Show on your graph where you would read off the solution to $\tan(kx) = 1$ if $x \in [-45°; 135°]$

3.3 Write down the period of g

3.4 Write down the amplitude of f

3.5 Write down the range of f

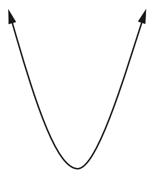
3.6 The graph of g is translated 2 units vertically. Write down the equation of the new graph.

3.7 The graph of f is translated 45° horizontally to the left, write down the equation of the new graph.

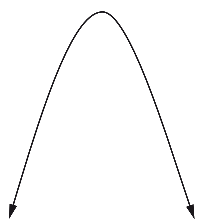
The graph of $y = a(x + p)^2 + q$

In Grade 10, you studied the graph of $y = ax^2 + q$.

If a is positive, i.e. $a > 0$, then the shape of the graph is ☺.



If a is negative, i.e. $a < 0$, then the shape of the graph is ☹.



The turning point of the graph of $y = ax^2 + q$ is $(0; q)$.

To find the x-intercepts, we let $y = 0$ and solve for x.

In grade 11, we must be able to sketch $y = a(x + p)^2 + q$.

The "a" value still determines whether the parabola smiles or frowns.

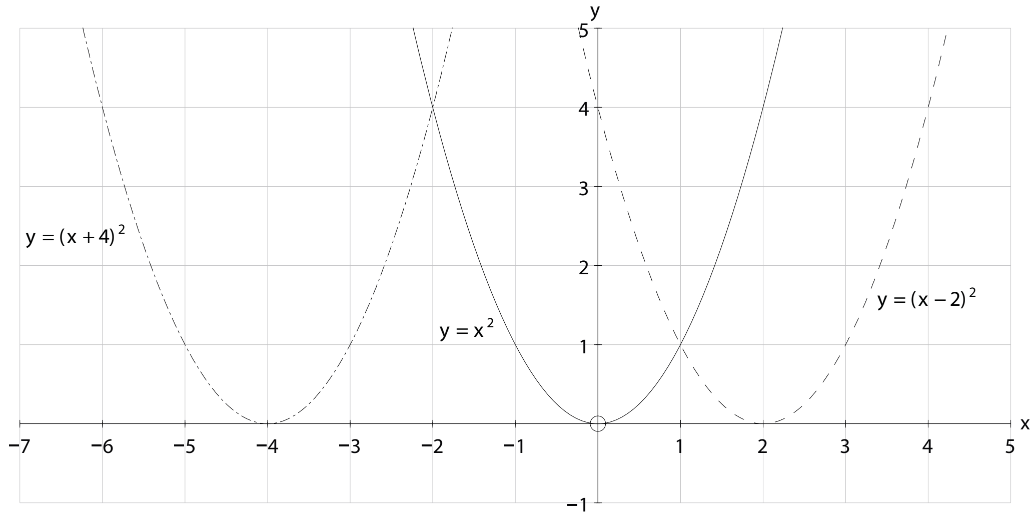
To find the x-intercepts we still let $y = 0$.

The coordinates of the turning point change from $(0; q)$ in $y = ax^2 + q$ to $(-p; q)$ in $y = a(x + p)^2 + q$.

As a result of adding p to the x before squaring it, the position of the turning point is no longer on the y-axis.

To find the y-intercept, we let $x = 0$.

Look at the graphs below.



In the graphs above, $y = (x - 2)^2$ is obtained by shifting $y = x^2$ horizontally to the right by 2 units. Also, the graph of $y = (x + 4)^2$ is obtained by shifting $y = x^2$ horizontally to the left by 4 units.

The domain for all three graphs is $(-\infty; \infty)$ or $\mathbb{R}$. The range for all three graphs is $(0; \infty)$.

Look at the graphs of $y = 2x^2 + 3$ and $y = 2(x - 1)^2 + 3$

The theory tells us:

Equation	$y = 2x^2 + 3$	$y = 2(x - 1)^2 + 3$
Turning Point	(0; 3)	(1; 3)
x-intercepts	none	none
y-intercept	(0; 3)	(0; 5)
Shape	☺	☺
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$[3; \infty)$	$[3; \infty)$

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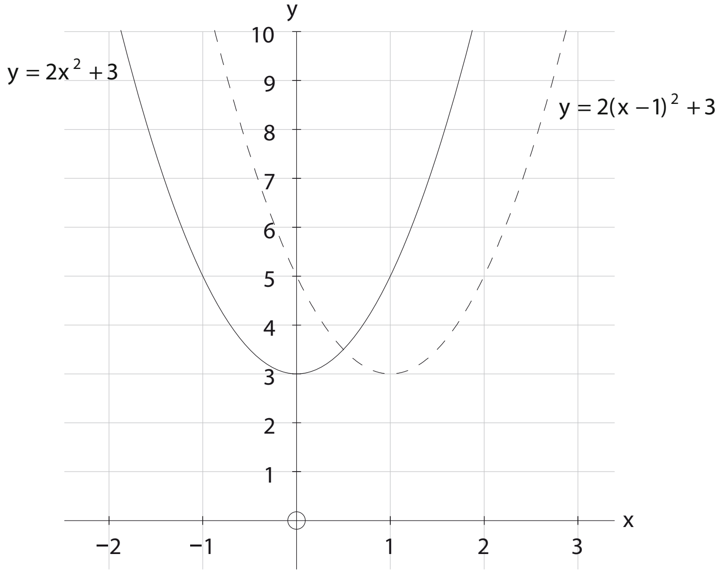
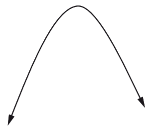


The graph of $y = 2(x - 1)^2 + 3$ is obtained by shifting horizontally the graph of $y = 2x^2 + 3$ to the right by 1 unit.

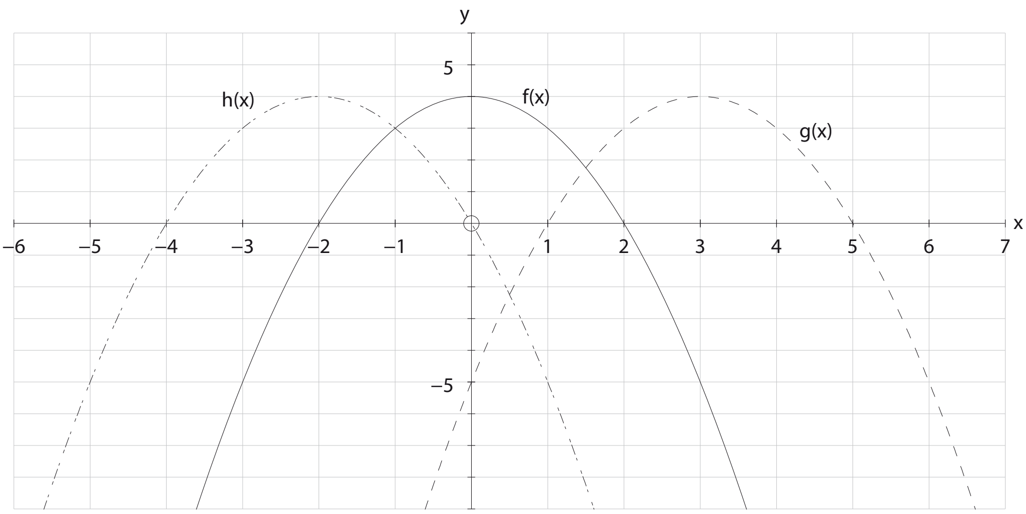
The next example shows us that if the shape of the parabola frowns, then



The same rules still apply, i.e.

- adding a positive p to x means a shift to the left
- adding a negative p to x means a shift to the right.

The graphs of $f(x) = -x^2 + 4$, $g(x) = -(x - 3)^2 + 4$ and $h(x) = -(x + 2)^2 + 4$ are drawn.



To draw the graph of $g(x) = -(x - 3)^2 + 4$, we simply shift the graph of $f(x) = -x^2 + 4$ horizontally to the right by 3 units.

To draw the graph of $h(x) = -(x + 2)^2 + 4$, we simply shift the graph of $f(x) = -x^2 + 4$ horizontally to the left by 2 units.

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SUMMARY

To sketch the graph of $y = a(x + p)^2 + q$,

1) Sketch the graph of $y = ax^2 + q$

2) Apply a horizontal shift to the right or left by p units depending on the sign of p

Domain = $\mathbb{R}$

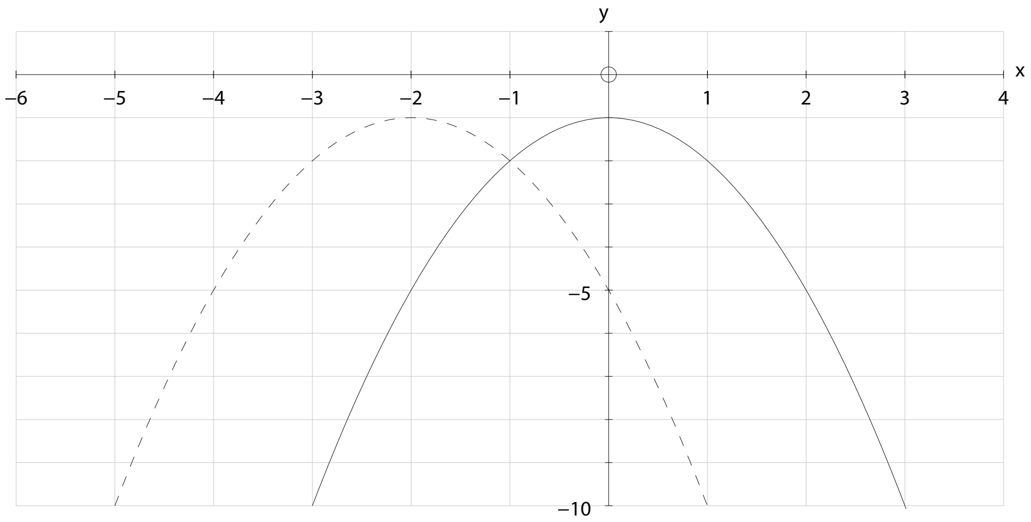
Range = $[q; \infty)$ if ☺ Range = $(-\infty; q]$ if ☹

Example

$y = -(x + 2)^2 - 1$ and $y = \frac{1}{2}(x - 3)^2 + 2$

Solution

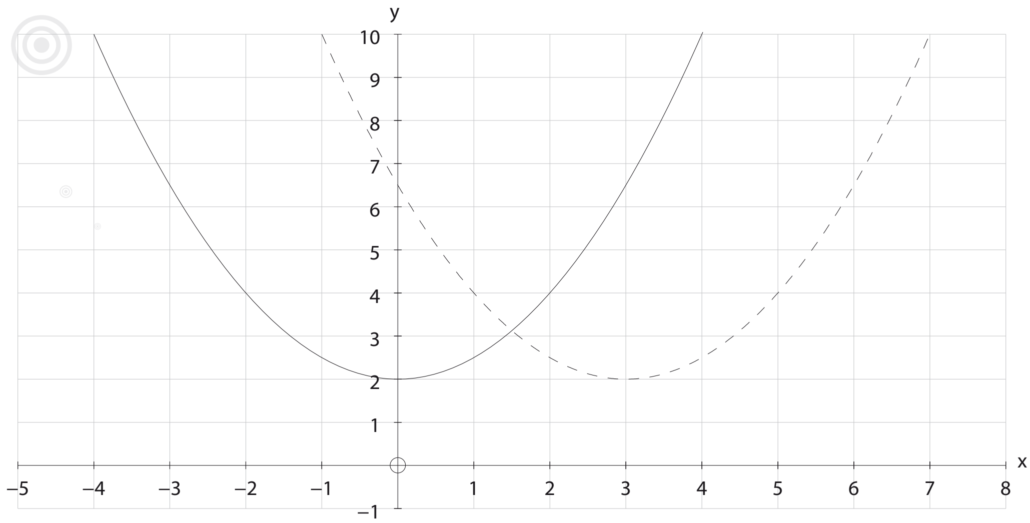
To sketch $y = -(x + 2)^2 - 1$, you must first look at the graph of $y = -x^2 - 1$. Then apply a horizontal shift of 2 units to the left.



It is important to label all intercepts with the axes. Note that the graph does not cut the x-axis, but the graph of $y = -(x + 2)^2 - 1$ cuts the y-axis at –5. To find the y-intercept on the y-axis, we let $x = 0$ in the equation.

$y = -(0 + 2)^2 - 1 = -4 - 1 = -5$

To sketch, $y = \frac{1}{2}(x - 3)^2 + 2$ you must first look at the graph of $y = \frac{1}{2}x^2 + 2$. Then apply a horizontal shift of 3 units to the right.



Note that the graph does not cut the x-axis, but the graph of $y = \frac{1}{2}(x-3)^2 + 2$ cuts the y-axis at $6\frac{1}{2}$. To find the y intercept on the y axis, we let x = 0 in the equation.

$y = \frac{1}{2}(0-3)^2 + 2 = \frac{1}{2}(9) + 2 = \frac{9}{2} + \frac{4}{2} = \frac{13}{2} = 6\frac{1}{2}$

IMPORTANT FACT

the turning point of $y = a(x+p)^2 + q$ is $(-p; q)$

Another example:

Sketch the graph of $f(x) = -2(x-4)^2 + 32$. From the equation, we get the following:

Shape	☹
Turning Point	(4; 32)
Range	$(-\infty; 32]$

To determine the y-intercept, we let $x = 0$

$y = -2(0-4)^2 + 32 = -2(16) + 32 = 0$

To determine the x-intercept, we let $y = 0$

$-2(x-4)^2 + 32 = 0$

$\therefore -2(x-4)^2 = -32$

$\therefore (x-4)^2 = 16$

$\therefore x - 4 = 4$ or $x - 4 = -4$

$\therefore x = 8$ or $x = 0$

Finally, the graph is drawn below. In the exam it is important that you label all turning points.

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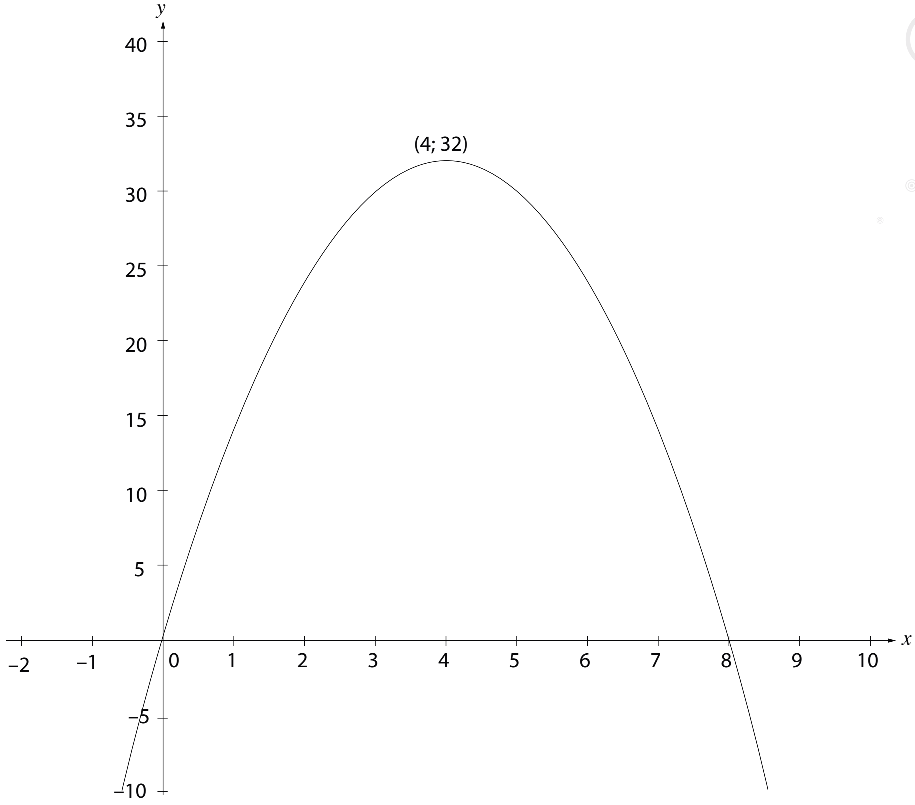


The graph of $y = ax^2 + bx + c$

Sometimes the equation of the parabola is not given in the form $y = a(x - p)^2 + q$. Instead the quadratic expression is given with the brackets expanded giving an equation of the form $y = ax^2 + bx + c$.

In this case one of two approaches can be used.

Approach 1: Complete the square to get the expression in the form $y = a(x - p)^2 + q$

Approach 2: Use $x = -\frac{b}{2a}$ to get the x-value at the turning point and then find the y-value at the turning point by substituting this x-value into the equation. The line $x = -\frac{b}{2a}$ is the axis of symmetry.

Let us look at examples:

Suppose you are asked to sketch $y = x^2 - 6x + 8$.

Approach 1: Complete the square

(Revise in Chapter on Quadratic Expressions)

$y = x^2 - 6x + 8$

$y = (x - 3)^2 - 9 + 8$

$y = (x - 3)^2 - 1$

Approach 2: Use $x = -\frac{b}{2a}$

$a = 1 \quad b = -6 \quad c = 8$

$x = -\frac{b}{2a}$

$\therefore x = -\frac{-6}{2(1)}$

$\therefore x = 3$

For the y-value,

$y = 3^2 - 6(3) + 8$

$y = 9 - 18 + 8 = -1$

Therefore the turning point is (3; –1)

The finding of x-intercepts and y-intercept remains the same.

x-intercepts: Let $y = 0$

$$x^2 - 6x + 8 = 0$$

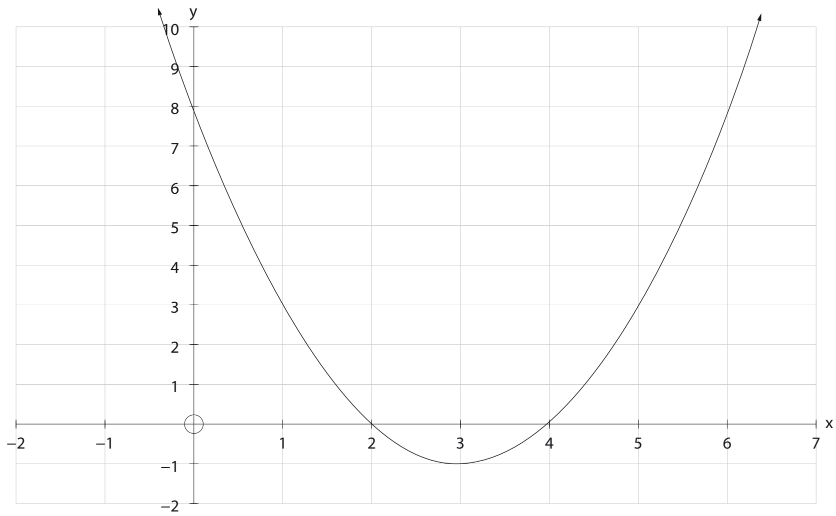
$$\therefore (x-4)(x-2) = 0$$

$$\therefore x = 4 \quad \text{or} \quad x = 2$$

y-intercept: Let $x = 0$

$$y = 0^2 - 6(0) + 8 = 8$$

The graph looks as follows:



Finding the equation of a parabola

Often the graph is drawn and the equation is required.

POSSIBILITY 1: Turning point given

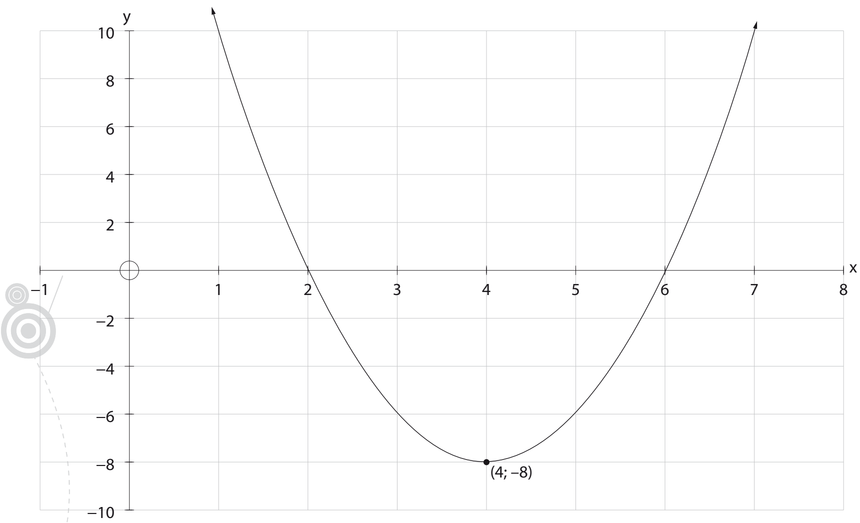
Step 1: Start with $y = a(x-p)^2 + q$

Substitute the coordinates of the turning point for p and q.

Step 2: Find the value of a by substituting any other point on the graph into the equation

Example

Example:



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Step 1: $y = a(x - 4)^2 - 8$

Step 2: Substitute (2; 0) or any other point into $y = a(x - 4)^2 - 8$

$0 = a(2 - 4)^2 - 8$

$\therefore 4a = 8$

$\therefore a = 2$

Therefore the equation is $y = 2(x - 4)^2 - 8$

POSSIBILITY 2:

The x-intercepts are given:

We find the equation by reversing the calculation for the x-intercepts.

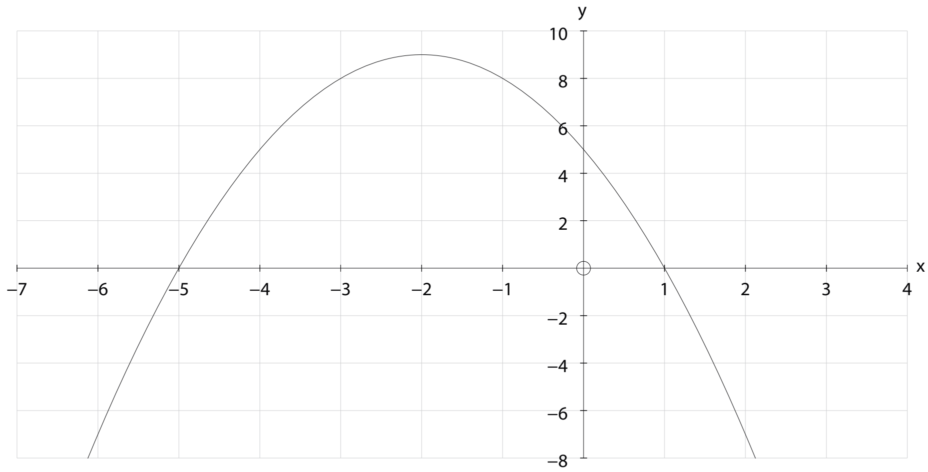
Step 1: Start with $y = a(x - x_1)(x - x_2)$

In this equation the x-intercepts are x_1 and x_2.

Step 2: Find a by substituting any point on the graph other than the x-intercept into the equation.

Example:

Example



Step 1: $y = a(x + 5)(x - 1)$. Notice how the signs of the x-intercepts change when you put them in the brackets

Step 2: Substitute (0; 5), or any other point on the parabola:

$5 = a(0 + 5)(0 - 1)$

$\therefore -5a = 5$

$\therefore a = -1$

Therefore the equation of the parabola is $y = -(x + 5)(x - 1)$.

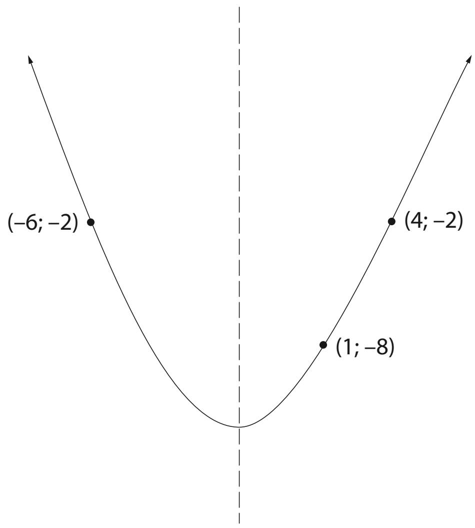
Sometimes it is necessary to multiply out the brackets.

Important fact about parabolas

The parabola is symmetrical about its axis of symmetry. It is often possible to find the axis of symmetry (x at turning point) by inspection.

Look at the example below:

The graph of $y = 2x^2 + bx + c$ is drawn below.



Determine the values of b and c.

The points (–6; –2) and (4; –2) are symmetrical to one another about the axis of symmetry (dotted line). Therefore the equation of the dotted line is $x = -1$. Exactly halfway between –6 and 4.

Therefore

Step 1: $y = 2(x + 1)^2 + q$. The a value is given in the question.

Step 2: We now will substitute (1; –8) to find the value of q.

$$y = 2(x + 1)^2 + q$$

$$\therefore -8 = 2(1 + 1)^2 + q$$

$$\therefore -8 = 8 + q$$

$$\therefore q = -16$$

The equation becomes $y = 2(x + 1)^2 - 16$. This time it is necessary to multiply the brackets out because of the way the question was asked.

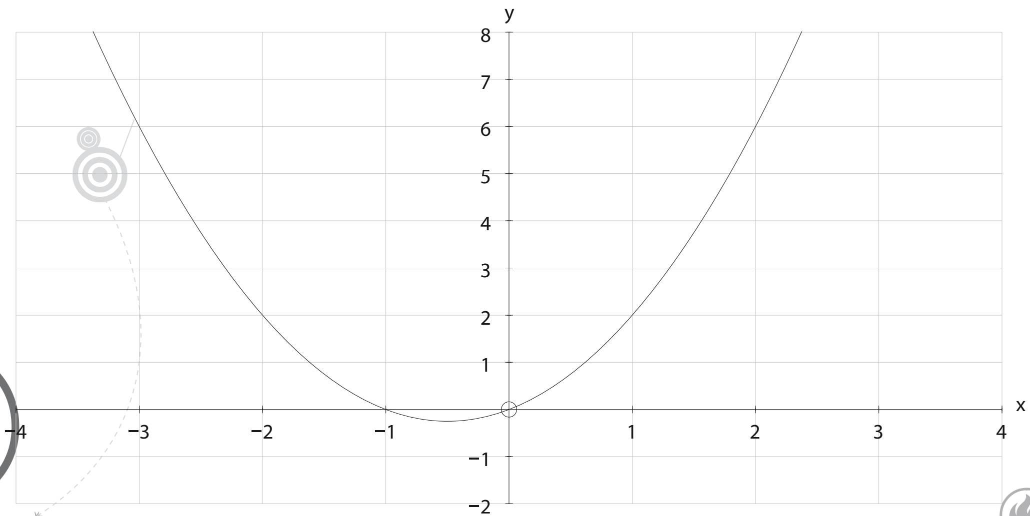
$y = 2(x + 1)^2 - 16 = 2(x^2 + 2x + 1) - 16 = 2x^2 + 4x + 2 - 16 = 2x^2 + 4x - 14$

Therefore $b = 4$ and $c = -14$

Activity

Activity 4

1. The graph of $f(x)$ is sketched. On separate systems of axes sketch the graph of



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1.1 $f(x+2)$

1.2 $f(x-3)$

1.3 $f(-x)$

1.4 $-f(x)$

2.1 Sketch the following graphs on the same set of axes

$f(x) = -x^2 + 4x - 6$ and $g(x) = 2(x-3)^2 - 8$

Show all intercepts with axes and turning points.

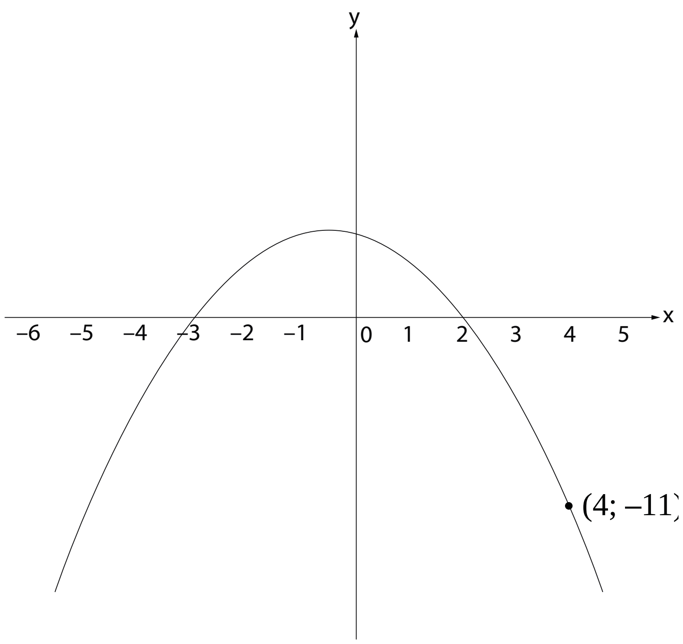
2.2 Find the x-values at the point(s) of intersection of f and g

2.3 Use your graph to solve

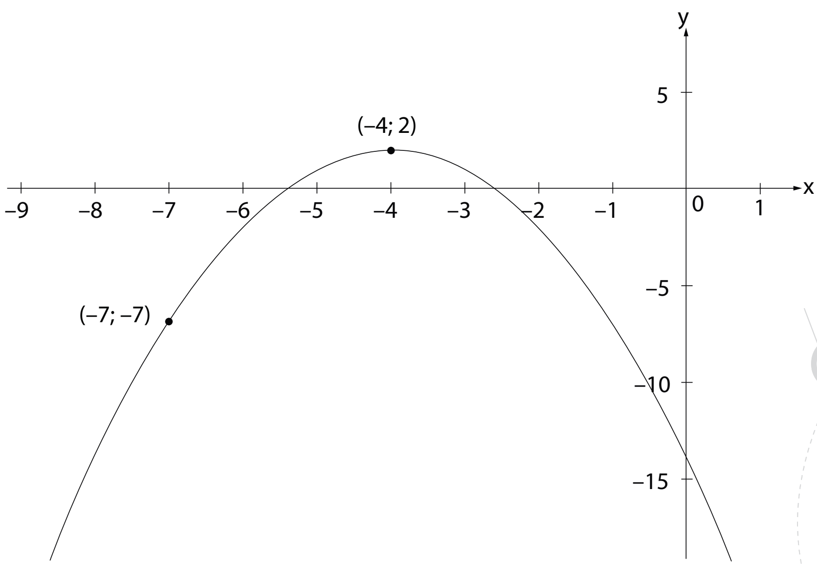
2.3.1 $f(x) < g(x)$ 2.3.2 $\frac{g(x)}{f(x)} > 0$

3. In each case the graph of $y = ax^2 + bx + c$ has been sketched. Determine the values of a, b and c.

3.1



3.2



The graph of $y = \frac{a}{x+p} + q$.

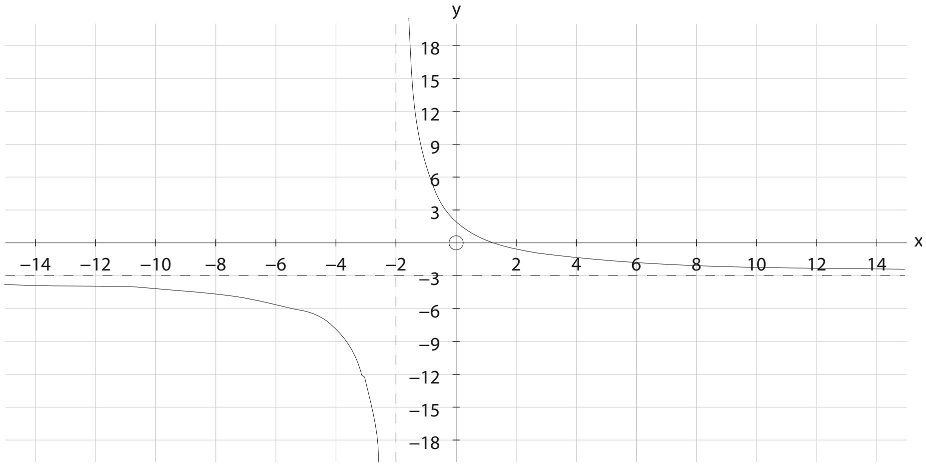
This is the equation of a hyperbola which has been shifted to the right or to the left by p units depending on the sign of p.

$\frac{a}{x+p} + q$ is undefined for $x = -p$.
Thus, the asymptotes are $x = -p$ and $y = q$.

To find the x-intercept, we let $y = 0$

To find the y-intercept, we let $x = 0$.

Look at the graph of $y = \frac{10}{x+2} - 3$ below,



The asymptotes are $x = -2$ and $y - 3$. (These are read from the equation).
x-intercept: Let $y = 0$:

$0 = \frac{10}{x+2} - 3 \therefore 10 = 3(x+2) \quad \therefore 3x = 4 \quad \therefore x = \frac{4}{3}$

y-intercept: Let $x = 0$: $y = \frac{10}{0+2} - 3 = 5 - 3 = 2$

Let us look at a second example, $y = -1 - \frac{8}{x-4}$.

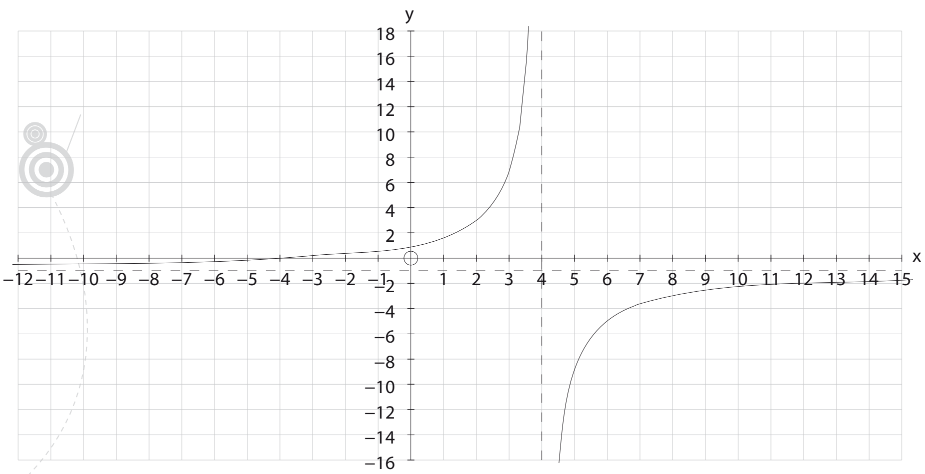
This equation can also be written as $y = \frac{-8}{x-4} - 1$.

Asymptotes: $x = 4$ and $y = -1$

y-intercept: Let $x = 0$: $y = -1 - \frac{8}{0-4} = -1 + 2 = 1$

x-intercept: Let $y = 0$: $0 = -1 - \frac{8}{x-4} \quad \therefore (x-4) = -8 \quad \therefore x = -4$

The hyperbola will look as follows:

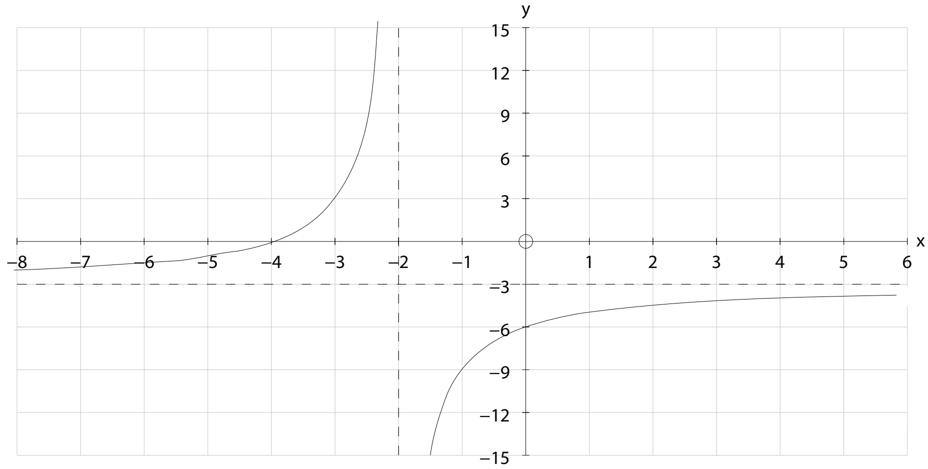


Determining the equation of a hyperbola when the graph is given

Step 1: The position of the asymptotes give us the value(s) of p and q in $y = \frac{a}{x+p} + q$.

Step 2: To find the value of a, we substitute any point on the graph into the equation.

EXAMPLE: The graph of $y = \frac{a}{x+p} + q$ is sketched below. Determine the value(s) of a, p and q.



From the graph we see that, $p = 2$ and $q = -3$.

The equation becomes: $y = \frac{a}{x+2} - 3$.

Substituting (0; –6) or any other point on the graph into the equation gives:

$-6 = \frac{a}{0+2} - 3 \quad \therefore -12 = a - 6 \quad \therefore a = -6$

The graph of $y = a \cdot b^{x+p} + q$

This is the graph of an exponential function.

This graph has one horizontal asymptote, i.e. $y = q$.

The easiest way to sketch this graph is to use your calculator:

Using the Casio fx-82 ES Plus,

Step 1: Mode 3: TABLE

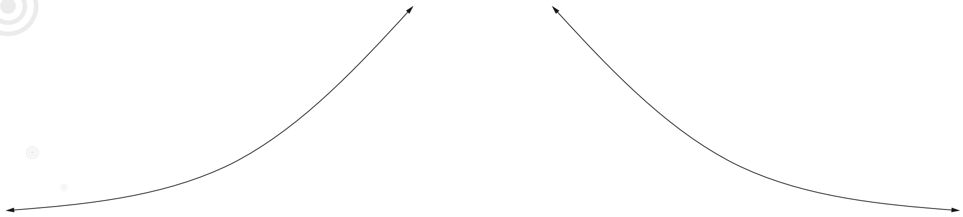
Step 2: Start? –1

Step 3: End? 1

Step 4: Step? 1

Step 5: Plot the points that the calculator returns and sketch the exponential through the points.

The exponential will be increasing OR it will be decreasing



To find the x-intercept, let $y = 0$.

Example

Example:

Sketch the graph of $y = 2\left(\frac{1}{2}\right)^{x+1} - 4$

Horizontal asymptote: $y = -4$

x-intercept: let $y = 0$:

$0 = 2\left(\frac{1}{2}\right)^{x+1} - 4 \therefore \left(\frac{1}{2}\right)^{x+1} = 2$

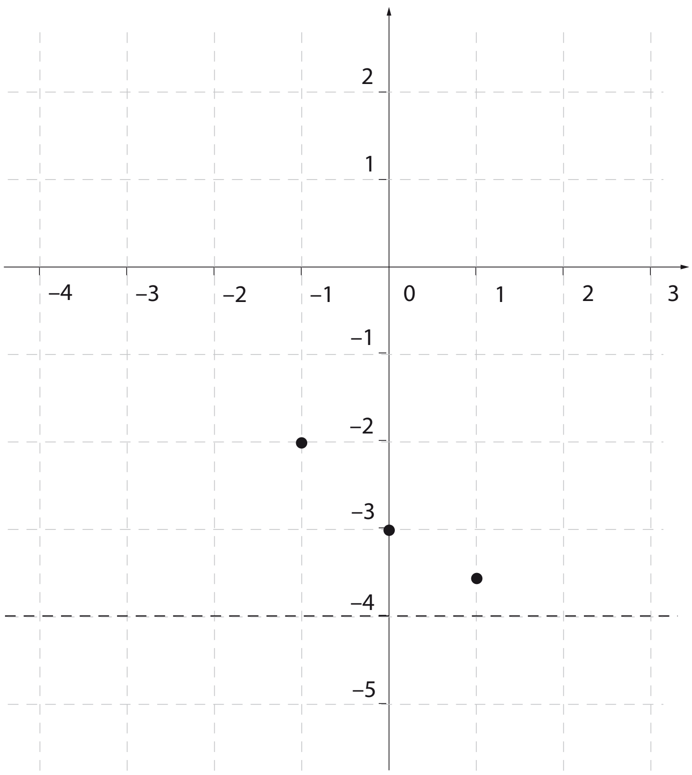
$\left(\frac{1}{2}\right)^{x+1} = \left(\frac{1}{2}\right)^{-1}$

$\therefore x + 1 = -1 \quad \therefore x = -2$

Using your calculator, we get the following table:

–1	–2
0	–3
1	–3,5

If we plot these points in the Cartesian plane as well as draw the horizontal asymptote, we will be able to see if the exponential is increasing or decreasing.



Now fit an exponential curve through the points so that it gets closer and closer to the asymptote.

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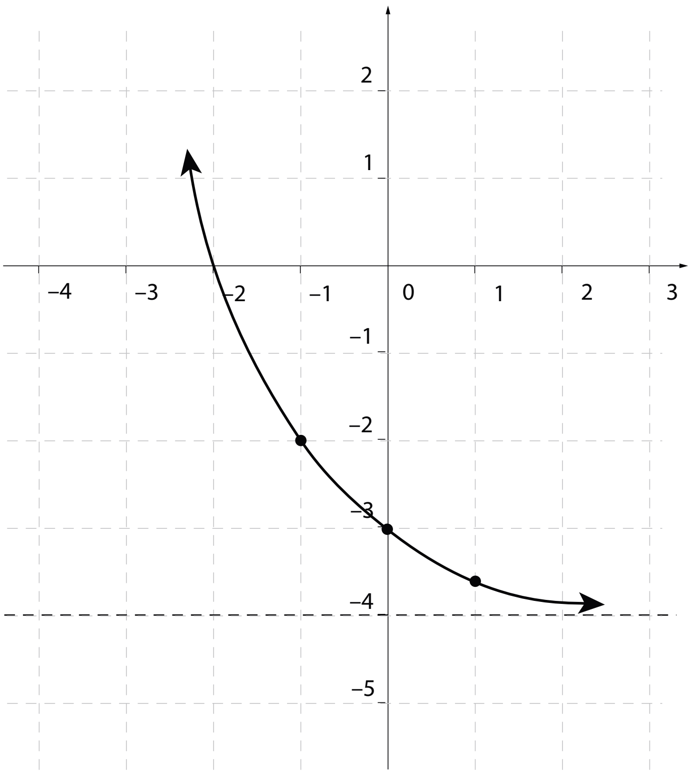


Let us look at a second example, $y = 3 \cdot (3)^{x-2} + 1$

Horizontal asymptote: $y = 1$

x-intercept: $0 = 3 \cdot (3)^{x-2} + 1$

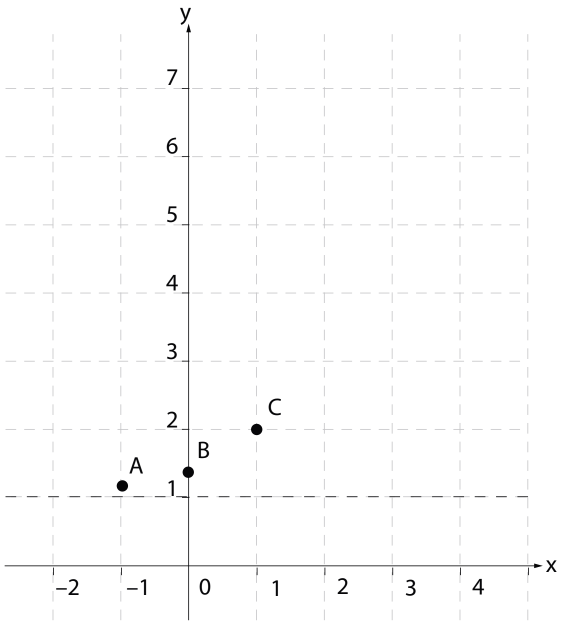
$\therefore\ 3(3)^{x-2} = -1$

$\therefore$ No x-intercept

Table of values from calculator:

−1	1,111
0	1,333
1	2

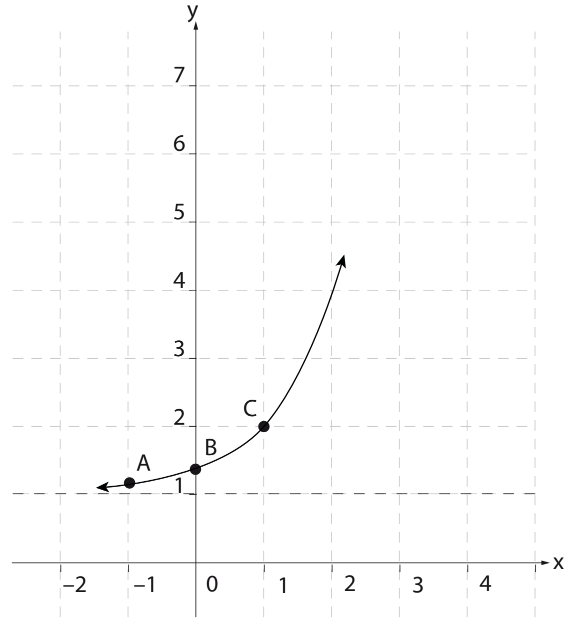
Now plot the points in the table as well as draw the asymptote:



Bb Bubblegum © 2009

It is now clear that the curve is an increasing curve approaching the asymptote from the left.

Now fit a smooth curve through the points.

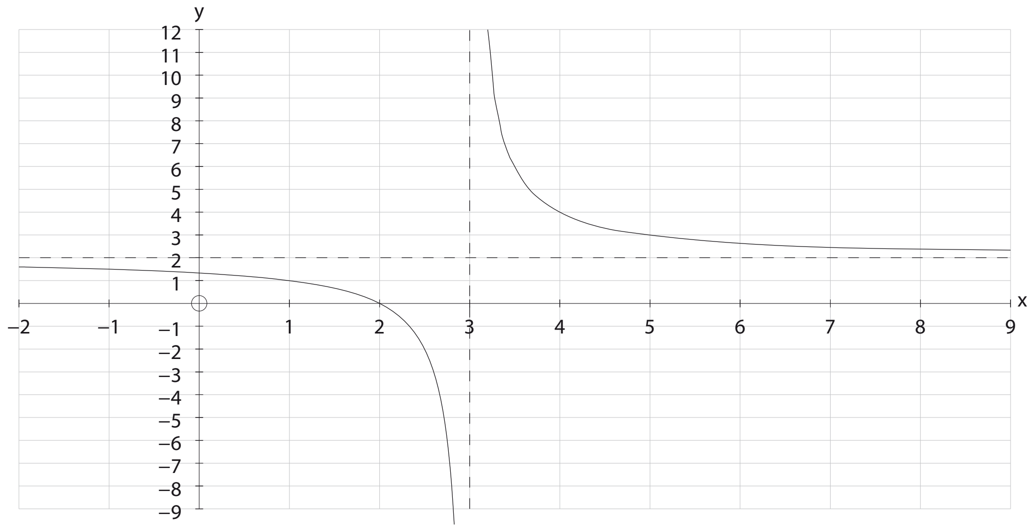


Note: for $b > 1$, $y = b^x$ is increasing and for $0 < b < 1$, $y = b^x$ is decreasing.

Activity

Activity 5

1. The sketch shows the graph of $y = f(x)$ which is a translation of the graph of $y = \frac{2}{x}$.



1.1 Write down the equation of f

1.2 The graph of f is shifted horizontally 3 units to the left. Write down the domain and range of the new graph.

2. The sketch shows the graph of $g(x) = a^{x+3} - b$. Determine the values of a and b.

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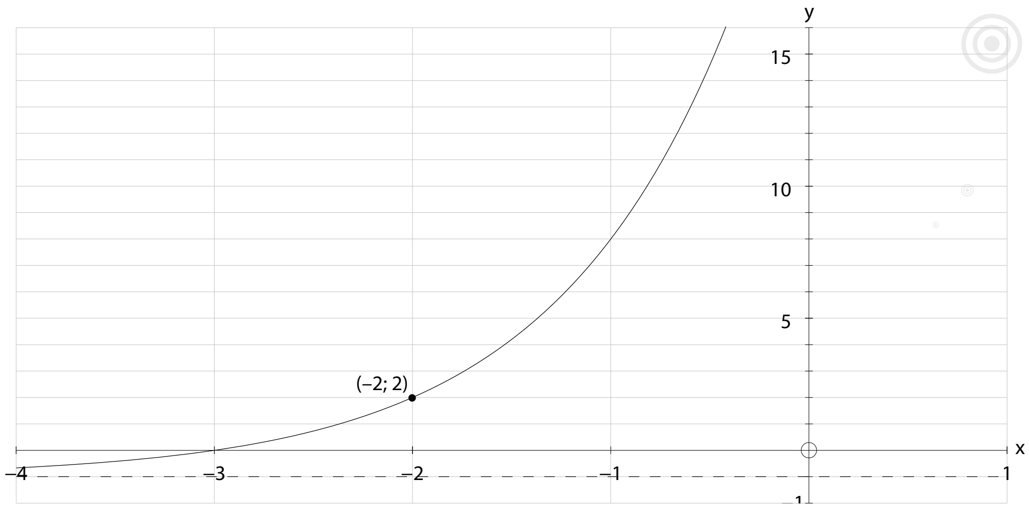
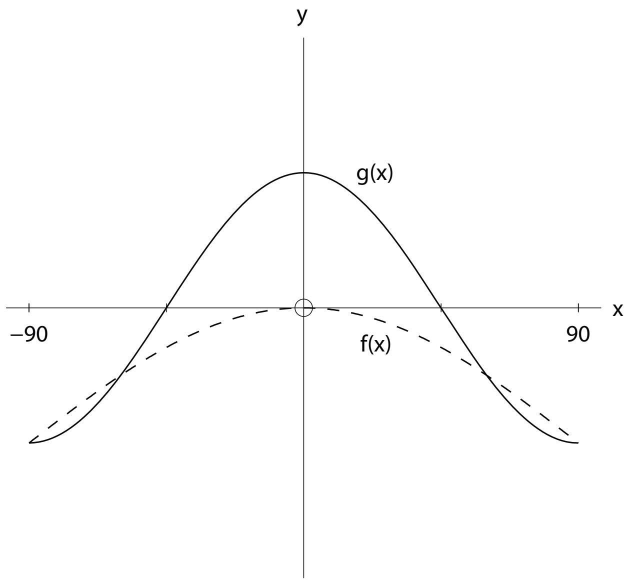


3. Given: $f(x) = 2^x$ and $g(x) = f(x + 2) + 1$

   Sketch the graph of $f(x)$.

4. The graphs of $f(x) = \cos x + p$ and $g(x) = \cos qx$ are sketched for $-90° \leq x \leq 90°$.



4.1 Write down the value of p if the graph of f touches the x-axis.

4.2 Write down the amplitude of f.

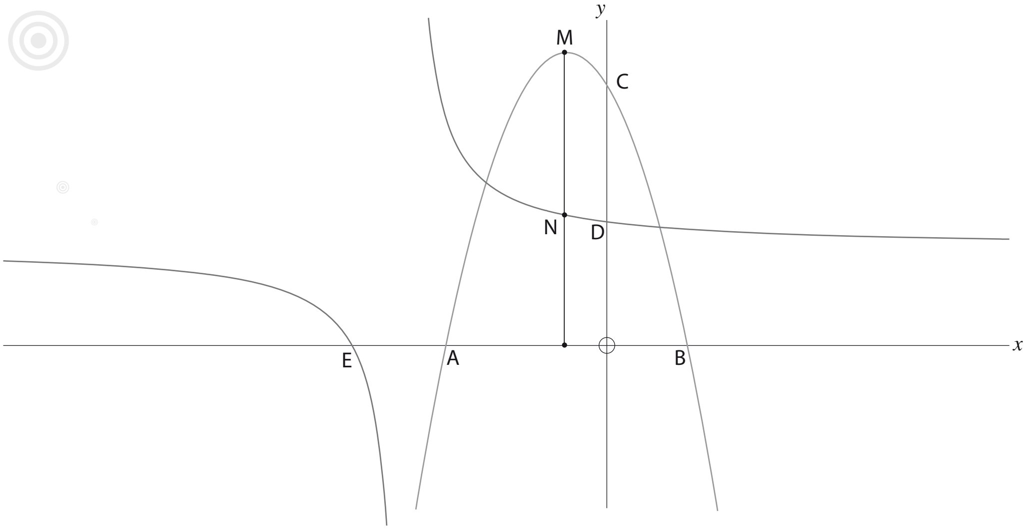
4.3 Write down the value of q if the period of g is half the period of f.

4.4 Write down the minimum value of f.

4.5 Use the graphs to solve: $g(x) \cdot f(x) \leq 0$ if $x \in [-90° ; 90°]$

5. The figure below represents the graphs of the following functions

   $f(x) = -2x^2 - 4x + 16$ and $g(x) = \dfrac{8}{x+5} + 6$



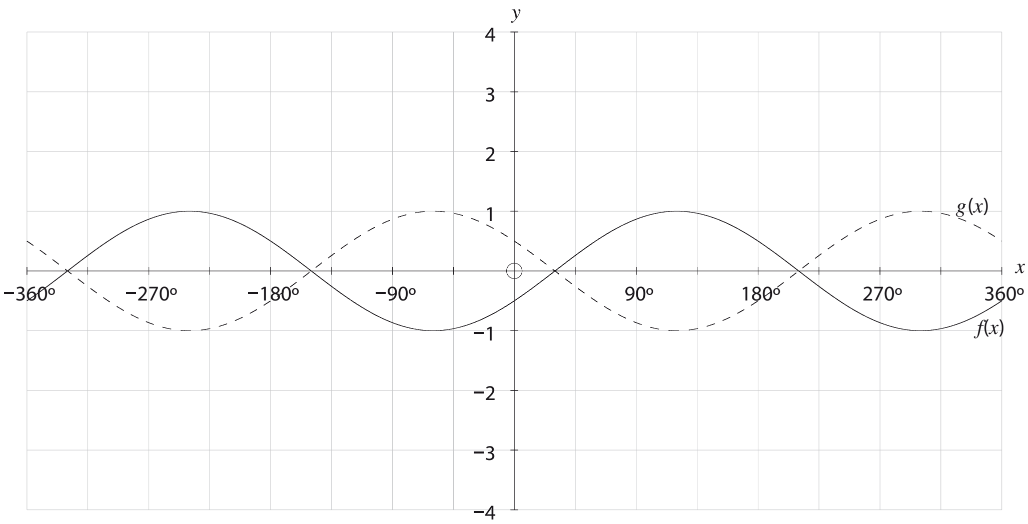
5.1 Determine the coordinates of A, B, C, D, E

5.2 If M is the turning point of $f(x)$, determine the length of MN

Solutions to Activities

Activity 1

1.1



1.2 The graphs are reflections of each other in the x-axis.

1.3 $\sin(x - 30°) = \cos(90° - (x - 30°)) = \cos(120° - x)$
$= -\cos(180° - (120° - x)) = -\cos(60° - x)$

$g(x) = \cos(x + 60°) = \sin(90° - (x + 60°))$

$= \sin(30° - x)$

$= -\sin(x - 30°)$

which is reflection in x axis of $f(x) = (x - 30°)$

Thus $f(x) = -g(x)$

2.1 $p = 20°$

2.2 $p = 25°$

2.3 $p = -45°$

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3.1 $a = 1$; $b = 2$ and $k = 45°$

3.2 360°

3.3 (135; –1)

3.4 $y = \cos(-180° + 45°) = \cos(-135°) = \cos 135° = -\cos 45° = -\frac{\sqrt{2}}{2}$

$\therefore E\left(-180°; -\frac{\sqrt{2}}{2}\right)$

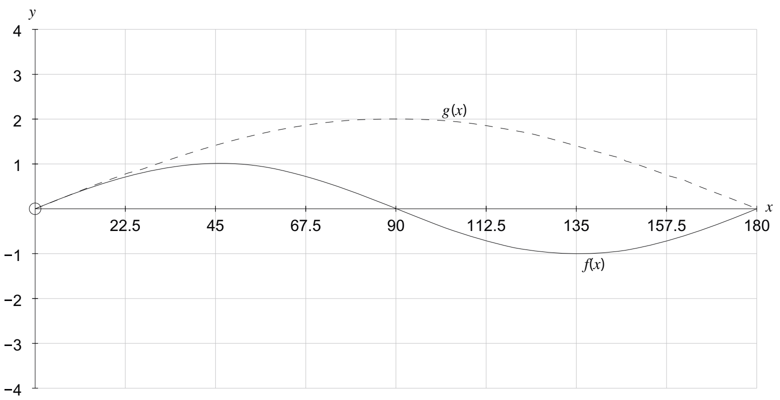
3.5 $f(x) = 2\sin(x + 20°)$

Activity 2

1. $\frac{360°}{3} = 120°$ 2. $\frac{360°}{2} = 180°$ 3. $\frac{180°}{\frac{1}{2}} = 360°$

Activity 3

1..1



1.2 $f \,|\, g = \{(0°; 0); (180°; 0)\}$

1.3.1 $90° < x < 180°$

1.3.2 No solution

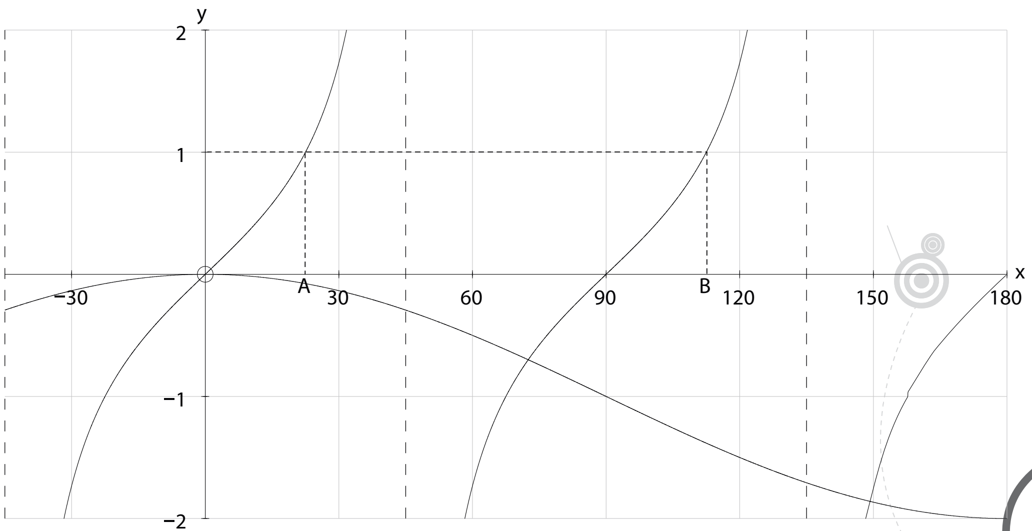
2.1 $k = 9$ Period = 40°

2.2 $k = \frac{1}{3}$ Period = 360° × 3 =1080° or 270° × 4

2.3 $k = 3$ Period = 60°

3.1 $a = 1$ $b = -1$ and $k = 2$

3.2 Shown at A and B



3.3 90°

3.4 1

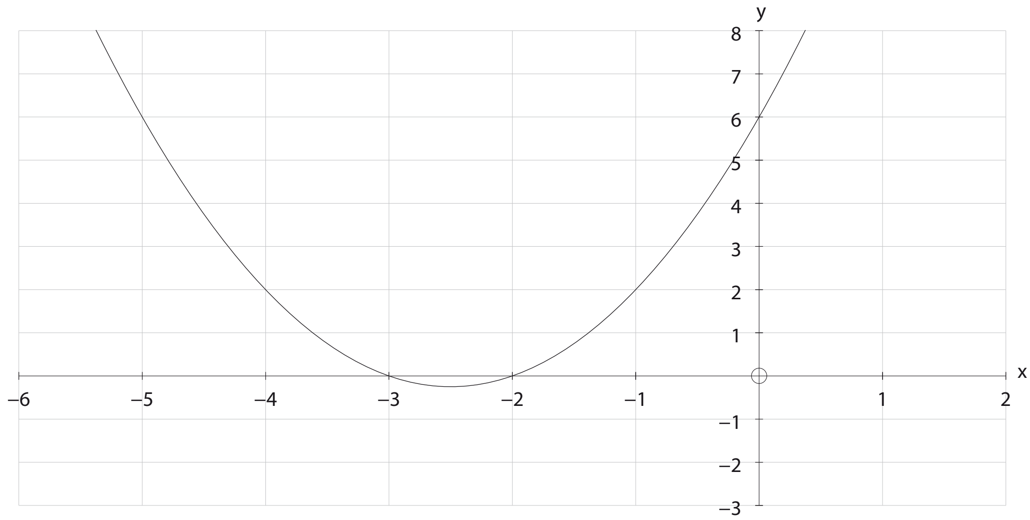
3.5 [–2; 0]

3.6 $y = \tan(2x) + 2$

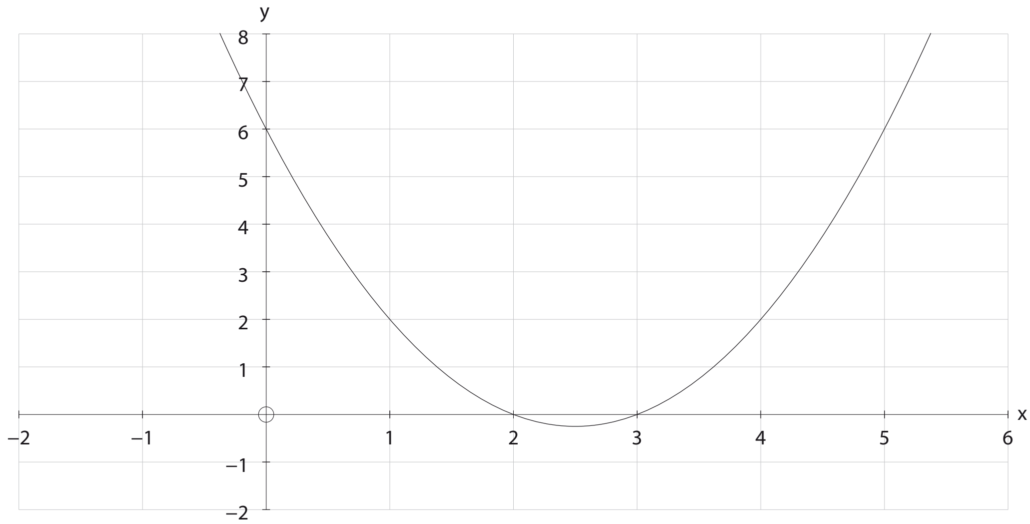
3.7 $y = \cos(x + 45°) - 1$

Activity 4

1.1 The original graph is shifted horizontally to the left by 2 units



1.2 The original graph is shifted horizontally to the right by 3 units



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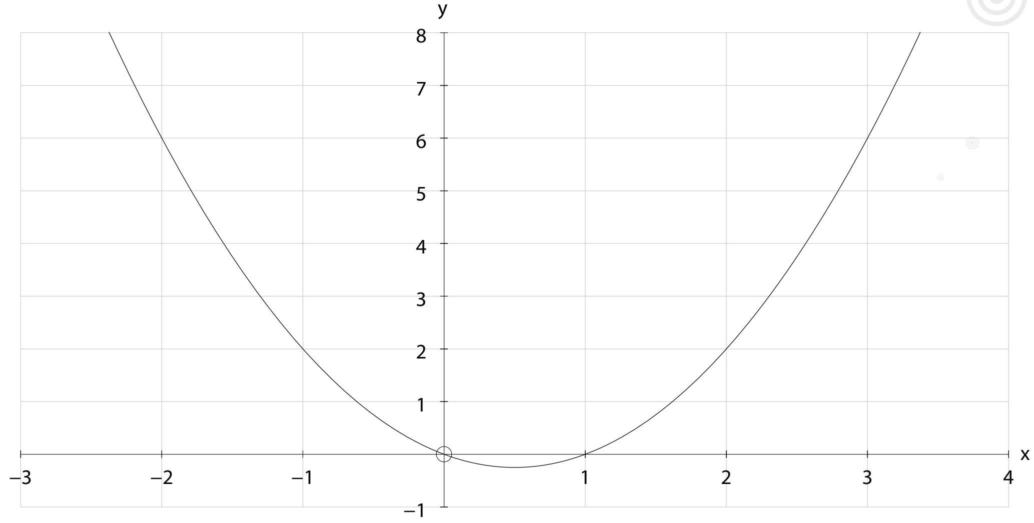
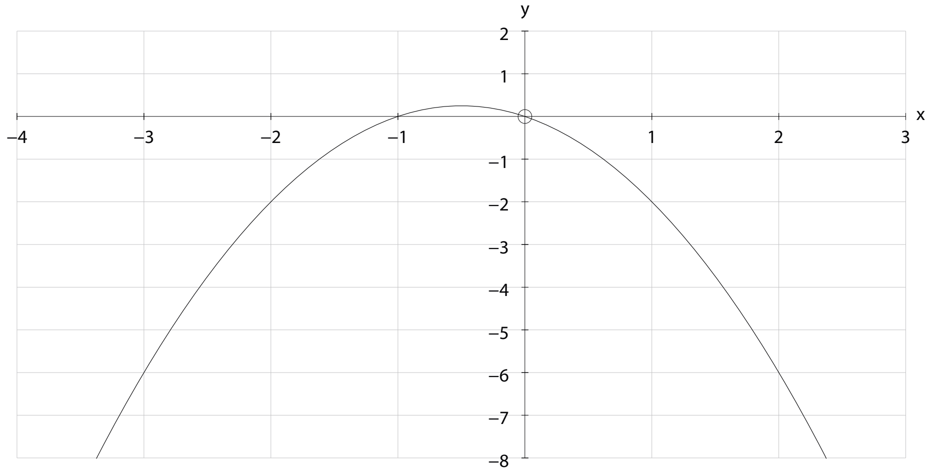
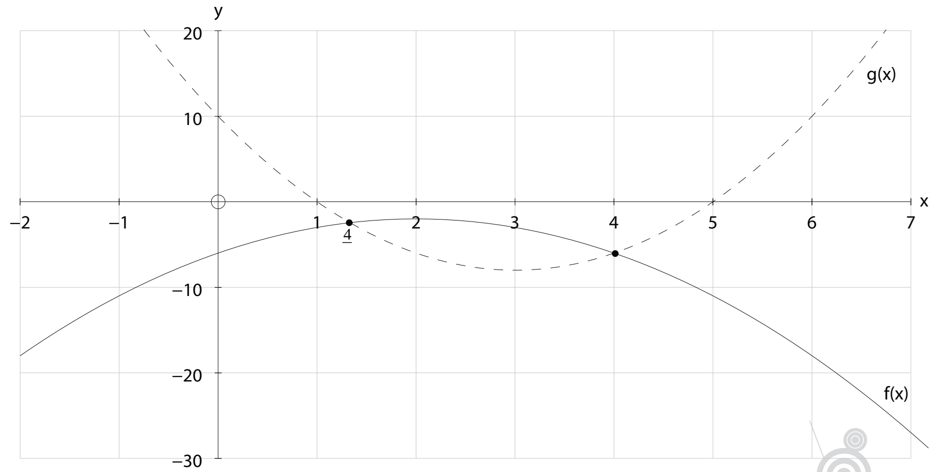
1.3 x is replaced with $-x$, therefore it is the graph symmetrical to the y-axis.



1.4 y is replaced with $-y$, therefore it is the graph symmetrical to the x-axis.



2.1



$f(x) = -x^2 + 4x - 6$ $g(x) = 2(x - 3)^2 - 8$

Shape: ☹ Shape: ☺

Turning point: Turning point:

$y = -(x - 2)^2 - 2$ (3; –8)

$\therefore$ (2; –2)

x-intercepts: none	x-intercepts: $2(x-3)^2 - 8 = 0$
	$\therefore (x-3)^2 = 4$
	$\therefore x - 3 = 2$ or $x - 3 = -2$
	$\therefore x = 5$ or $x = 1$
y – intercept: $y = -6$	y intercept: $y = 2(0-3)^2 - 8 = 10$
Domain = $\mathbb{R}$	Domain = $\mathbb{R}$
Range = $(-\infty; -2]$	Range = $[-8; \infty)$

2.2 For point(s) of intersection, we equate the two quadratic expressions:

$$-x^2 + 4x - 6 = 2(x-3)^2 - 8$$

$$\therefore -x^2 + 4x - 6 = 2(x^2 - 6x + 9) - 8$$

$$\therefore -x^2 + 4x - 6 = 2x^2 - 12x + 18 - 8$$

$$\therefore 3x^2 - 16x + 16 = 0$$

$$\therefore (3x-4)(x-4) = 0$$

$$\therefore x = \frac{4}{3} \text{ or } x = 4$$

2.3.1 $x < \frac{4}{3}$ or $x > 4$

2.3.2 Since $f(x) < 0$ for all values of x, it follows that $g(x) < 0$.

Therefore, $1 < x < 5$

3.1 $y = a(x+3)(x-2)$

Substituting $(4; -14)$, we get

$-14 = a(4+3)(4-2)$

$\therefore a = -1$

$y = -(x+3)(x-2)$

$\therefore y = -(x^2 + x - 6)$

$\therefore y = -x^2 - x + 6$

$\therefore a = -1 \quad b = -1 \quad c = 6$

3.2 $y = a(x+4)^2 + 2$

substituting $(-7; -7)$, we get

$-7 = a(-7+4)^2 + 2$

$\therefore 9a = -9$

$\therefore a = -1$

Therefore, the equation is

$y = -1(x+4)^2 + 2$

$= -1(x^2 + 8x + 16) + 2$

$= -x^2 - 8x - 14$

$a = -1 \quad b = -8 \quad c = -14$

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Activity 5

1.1 $f(x) = \frac{2}{x-3} + 2$

1.2 Domain = $\mathbb{R}-\{0\}$

Range = $\mathbb{R}-\{2\}$

2. $b = 1$

Substituting (–2; 2) or any other point on the graph into $y = a^{x+3} - 1$, we get:

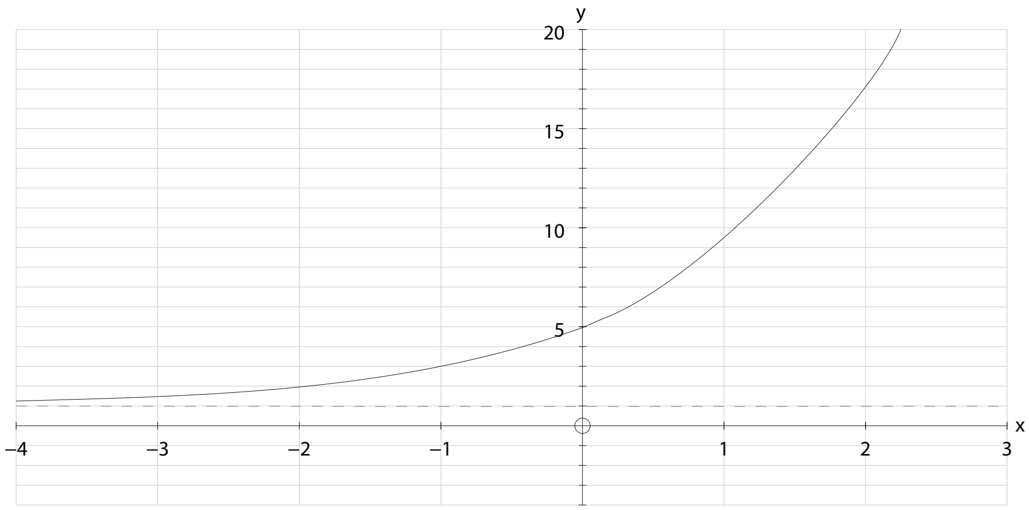
$2 = a^{-2+3} - 1 \quad \therefore\ a = 3$

3. $g(x) = 2^{x+2} + 1$

Asymptote: $y = 1$

x-intercept: $0 = 2^{x+2} + 1 \quad \therefore\ 2^{x+2} = -1$

Therefore no x-intercepts



4.1 $p = -1$

4.2 1

4.3 $q = 2$

4.4 –2

4.5 Since $f(x) < 0$, it follows that $g(x) > 0$.

Thus, $-45° \leq x \leq 45°$

5.1 A and B : x-intercepts of f:

$-2x^2 - 4x + 16 = 0$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = -4 \quad \text{or} \quad x = 2$

$\therefore$ A(–4; 0) B(2; 0)

C is the y-intercept of f: let $x = 0$

C(0; 16)

D is the y-intercept of g: Let $x = 0$, we get $y = \frac{8}{0+5} + 6 = 7{,}6$ $\therefore$ D (0; 7,6)

E is the x-intercept of g: Let $y = 0$, we get

$0 = \frac{8}{x+5} + 6$

$\therefore\ 0 = 8 + 6(x + 5)$

$\therefore\ 0 = 8 + 6x + 30$

$\therefore\ 6x = -38$

$\therefore\ x = -\frac{19}{3}$

$\therefore\ E\left(-\frac{19}{3}; 0\right)$

5.2 $f(x) = -2x^2 - 4x + 16$

$\therefore\ f(x) = -2(x + 1)^2 + 18$

$\therefore\ M(-1; 18)$

Thus we know that

$x_N = -1$

$\therefore\ y = \frac{8}{-1 + 5} + 6 = 8$

Thus N(–1; 8)

Therefore MN = 10 units.(18 – 8 = 10)

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