LESSON 6

FACTORING AND SOLVING QUADRATIC EQUATIONS

Solving the equation $ax^2 + bx + c = 0$

Example

Example 1

Solve the equation $x^2 + 8x + 15 = 0$.

The zero principle: When we solve quadratic equations, it is essential to get the one side of the equation equal to zero before we factorise the quadratic part of the equation. The zero principle is very important, since when the quadratic is factorised, it will form a product of the two factors that will be equal to zero. So now we see that when factoring we obtain:

$$x^2 + 8x + 15 = 0$$

$$\therefore (x + 3)(x + 5) = 0$$

Now to complete the problem, we simply apply a very basic principle that applies to a product that is equal to zero. If we have two factors A and B such that A·B = 0, then only one of the two has to be zero for the product to be zero. That would mean that A = 0 or B = 0.

So now:

$$(x + 3)(x + 5) = 0$$

$$\therefore x = -3 \text{ or } x = -5$$

Note that if this product A·B = 6, then we cannot be certain of the values of each of the factors A and B since $6 = 2 \times 3$ or $6 = 1 \times 6$ or $6 = \frac{1}{2} \times 12$, etc. So it is important that we ensure the one side is zero.

Example

Example 2

Fully factorise $3x^2 + 33x + 84$ and hence solve $3x^2 + 33x + 84 = 0$.

To factorise:

$$3x^2 + 33x + 84 = 3(x^2 + 11x + 28)$$
$$= 3(x + 4)(x + 7)$$

When solving $3x^2 + 33x + 84 = 0$ the zero principle applies. So:

$$3(x + 4)(x + 7) = 0$$

$$\therefore (x + 4)(x + 7) = 0 \qquad \text{(We can divide both sides by 3)}$$

$$\therefore x = -4 \text{ or } x = -7$$

Activity 1

Activity

Solve for x in each of the following:

1. $x^2 - 7x + 10 = 0$

2. $x^2 + 4x = 21$

3. $12x^2 - 6 = -17x - 12$

4. $x^2 = 6 - 5x$

5. $6x^2 + 11x + 3 = 0$

6. $8x^2 - 14x + 3 = 0$

7. $21x^2 = 41x - 10$

8. $3x^2 + x = 14$

Solving quadratic equations using the quadratic formula

If we cannot find the factors for a quadratic equation of the form $ax^2 + bx + c = 0$, then it is quite possible that the roots are irrational.

So if $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We prove this by colmpeting the square:

$ax^2 + bx + c = 0$

$(\div a)$: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Complete the square:

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$$

$$\therefore (x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$\therefore x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = 0$

(x $4a$): $4a^2x^2 + 4abx + 4ac = 0$

Complete the square:

$$4a^2x^2 + 4abx^2 + b^2 = b^2 - 4ac$$

$$\therefore (2ax + b)^2 = b^2 - 4ac$$

$$\therefore 2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$\therefore 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You may be required to give the solutions for x correct to a specific number of decimal places, or alternatively in simplest surd form.

Although you can use the formula as an alternative to factorising an equation and then solving, make sure you can factorise quadratic expressions, as the next step will be factoring cubic polynomials. The formula only works on quadratic equations, and not on cubics. (More about this in Grade 12).

Example

Example 1

Solve for x in $2x^2 + x - 2 = 0$

Remember the standard form

$ax^2 + bx + c = 0$

So: $a = 2$; $b = 1$; $c = -2$

$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1 - 4(-4)}}{4}$

$= \frac{-1 \pm \sqrt{17}}{4}$

So in simplest surd form:

$\therefore\ x = \frac{-1 + \sqrt{17}}{4}$ or $\frac{-1 - \sqrt{17}}{4}$

Correct to two decimal places:

$x = 0{,}78 \ or \ x = -1{,}28$

Example 2

Solve for x in $(x + 3)(2 - x) = 7x$.

We need to write the equation in standard form:

$(x + 3)(2 - x) = 7x$

$\therefore\ 2x - x^2 + 6\ -\ 3x - 7x = 0$

$\therefore\ -x^2 - 8x + 6 = 0$

$\therefore\ x^2 + 8x - 6 = 0$

Now $a = 1$ $b = 8$ $c = -6$

$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-8 \pm \sqrt{64 - 4(-6)}}{2}$

$= \frac{-8 \pm \sqrt{88}}{2}$

$= \frac{-8 \pm 2\sqrt{22}}{2}$

$= \frac{2(-4 \pm \sqrt{22})}{2}$

$= -4 \pm \sqrt{22}$

Using the casio Fx 82 - ES Plus: If you enter this line on your calculator, the answer that you get will be in simplest surd form.

So in simplest surd form:

$\therefore\ x = -4 - \sqrt{22} \ or \ x = -4 + \sqrt{22}$

Correct to two decimal places:

$x = -7{,}32 \ or \ x = -0{,}69$

Activity

Activity 2

Solve for x in each of the following by using the formula. Leave your answers in simplest surd form and rounded to two decimal places where applicable:

1. $2x - 5 = \frac{7}{x}$

2. $\frac{1}{2}x^2 - 3x + 0,7 = 0$

3. $4x^2 - 20x + 9 = 0$ 4. $2x^2 + 4x + 3 = 0$

Solving quadratic equations by completing the square

We can also solve equations of the form $ax^2 + bx + c = 0$ by completing the square, which means rewriting the equation in the form $a(x-p)^2 + q = 0$. Do you notice that the x is contained in a bracket which is under a square. So it will be easier to solve as we can merely take the square root both sides.

Equations with squares

Example 1

Example

Solve for x in $4(x-3)^2 = 16$

We divide to isolate the bracket:

$(x-3)^2 = \frac{16}{4}$

$\therefore (x-3)^2 = 4$

Now square root both sides and make sure not to forget the " $\pm$ " sign:

$\therefore x - 3 = \pm 2$

And solve:

$\therefore x = 3 \pm 2$

$\therefore x = 5 \text{ or } x = 1.$

Example 2

Example

Solve for x in $4(x-3)^2 = 5$

We divide to isolate the bracket:

$(x-3)^2 = \frac{5}{4}$

Now square root both sides and make sure not to forget the " $\pm$ " sign:

$\therefore x - 3 = \pm\frac{\sqrt{5}}{2}$

And solve:

$\therefore x = 3 \pm \frac{\sqrt{5}}{2}$ in surd form

or rounded to two decimal places:

$x = 1{,}88 \text{ or } x = 4{,}12$

Example

Example 3

Solve for x in $4(x-3)^2 = -5$

Here it is clear to see that the left hand side of the equation which is $4(x-3)^2$ is always positive. This happens because anything squared is positive. So $(x-3)^2 \geq 0$. If we multiply this by a positive number, it remains positive. So $4(x-3)^2 \geq 0$. So the right hand side of the equation should be positive, which it is not. So this equation has no real solutions. We say that it has non-real roots since solving it would mean, therefore, that $x - 3 = \pm \frac{\sqrt{-5}}{2}$ and we know that $\sqrt{-5}$ is a non-real number.

It is clear to see that if the x is contained in a bracket, we can say a lot about the value of the expression, and we can also solve it more easily. So let us look at a few methods that help us complete the square.

Completing the square to solve the equation $ax^2 + bx + c = 0$

Example

Example 1

In the following example we will solve the equation $x^2 - 2x - 24 = 0$ by completing the square.

We will do two different methods, and you can choose which one to follow:

Method 1	**Method 2**
Solve for x in $x^2 - 2x - 24 = 0$	Solve for x in $x^2 - 2x - 24 = 0$
To complete the square we must follow the steps below: **1. Ensure that a in $ax^2 + bx + c = 0$ is equal to one** **2. Move the constant term(s) to the opposite side of the equation.** **3. To complete the square, we add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.** **4. Factorise and isolate the completed square.** **5. Take the square root of both sides.** **6. Isolate x and solve.**	To complete the square we must follow the steps below: **1. Move the constant term to the opposite side of the equation.** **2. Take the value of a and multiply it by 4 so you create $4a$.** **3. Multiply each term by $4a$.** **4. Calculate b^2 and add it to both sides (note: the b value that you use is determined from the original equation).** **5. Factorise the expression (in the form of a perfect square).** **6. Take the square root both sides.** **7. Isolate x and solve.**

In this equation, $a = 1$. So we move to step two where we calculate $\left(\frac{b}{2}\right)^2$. $b = -2$, so $\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = 1$. Now we add this to each side of the equation and move the –24 to the right hand side of the equation: $x^2 - 2x + 1 = 24 + 1$. Now the square is completed and we can factorise: $\therefore x^2 - 2x + 1 = 24 + 1$ $\therefore (x - 1)^2 = 25$ $\therefore x - 1 = \pm 5$ $\therefore x = 1 \pm 5$ $\therefore x = 6 \text{ or } x = -4$.	We isolate the constant term to get $x^2 - 2x = 24$. In this equation $a = 1$, so $4a = 4$. So if we multiply each term by 4 we get: $4x^2 - 8x = 96$ Now if we add $b^2 = (-2)^2 = 4$ to both sides: $4x^2 - 8x + 4 = 96 + 4$. The square is completed and we can solve: $\therefore 4x^2 - 8x + 4 = 96 + 4$ $\therefore (2x - 2)^2 = 100$ $\therefore 2x - 2 = \pm 10$ $\therefore 2x = 2 \pm 10$ $\therefore x = 1 \pm 5$ $\therefore x = 6 \text{ or } x = -4$

You can choose which method is the easier for you to use. Always remember that working with an expression $(y = ax^2 + bx + c)$ is different to working with an equation $(ax^2 + bx + c = 0)$.

Example

Example 2

Solve for x if $2x^2 - 3x - 1 = 0$ by completing the square:

Method 1	**Method 2**
Move the constant: $2x^2 - 3x = 1$ $a = 2$ so we divide by 2 throughout: $x^2 - \frac{3}{2}x = \frac{1}{2}$ $b = -\frac{3}{2}, \textit{ so } \left(\frac{b}{2}\right)^2 = \left(-\frac{\frac{3}{2}}{2}\right)^2 = \left(-\frac{3}{4}\right)^2 = \frac{9}{16}$ We add this to both sides of the equation: $x^2 - \frac{3}{2}x + \frac{9}{16} = \frac{1}{2} + \frac{9}{16}$ Now we solve for x: $x^2 - \frac{3}{2}x + \frac{9}{16} = \frac{1}{2} + \frac{9}{16}$ $\therefore \left(x - \frac{3}{4}\right)^2 = \frac{8+9}{16} = \frac{17}{16}$ $\therefore x - \frac{3}{4} = \pm\frac{\sqrt{17}}{4}$ $\therefore x = \frac{3 \pm \sqrt{17}}{4}$ Correct to two decimals: $\therefore x = 1{,}78 \textit{ or } x = -0{,}28$	Move the constant: $2x^2 - 3x = 1$ Since $a = 2$ we see that $4a = 8$. Since $b = -3$, we have $b^2 = (-3)^2 = 9$. So first multiply all terms by $4a$, and then add the 9 to both sides: (multiply by 8): $16x^2 - 24x = 8$ Add 9: $16x^2 - 24x + 9 = 8 + 9$ Now solve for x: $\therefore (4x - 3)^2 = 17$ $\therefore 4x - 3 = \pm\sqrt{17}$ $\therefore x = \frac{3 \pm \sqrt{17}}{4}$ Correct to two decimals: $\therefore x = 1{,}78 \textit{ or } x = -0{,}28$

Example

Example 3

Solve for x if $3x^2 - 6x - 2 = 0$ by completing the square:

Method 1	**Method 2**
Isolate the constant term and divide by 3: $3x^2 - 6x = 2$ $\therefore x^2 - 2x = \frac{2}{3}$ Complete the square and solve: $x^2 - 2x = \frac{2}{3}$ $\therefore x^2 - 2x + 1 = \frac{2}{3} + 1 \quad \left[\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = 1\right]$ $\therefore (x - 1)^2 = \frac{5}{3}$ $\therefore x = 1 \pm \sqrt{\frac{5}{3}}$ (surd form)	Isolate the constant term, and multiply by $4a$. Then add b^2 to both sides: $3x^2 - 6x = 2 \qquad (4a = 12; b^2 = 36)$ $\therefore 36x^2 - 72x + 36 = 24 + 36$ $\therefore (6x - 6)^2 = 60$ $\therefore 6x - 6 = \pm\sqrt{60}$ $\therefore x - 1 = \pm\frac{\sqrt{60}}{6}$ $\left(\frac{\sqrt{60}}{6} = \frac{\sqrt{6} \times \sqrt{10}}{6} = \frac{\sqrt{6} \times \sqrt{10}}{\sqrt{6} \times \sqrt{6}} = \sqrt{\frac{10}{6}} = \sqrt{\frac{5}{3}}\right)$ $\therefore x = 1 \pm \sqrt{\frac{5}{3}}$

Activity

Activity 3

Solve for x in each of the following leaving your answers in simplest surd form and rounded to two decimal places where applicable: (use either of the methods for completing the square)

1. $x^2 - 6x + 3 = 0$

2. $x^2 - 4x + 5 = 0$

3. $2x^2 - x - 5 = 0$

4. $3x^2 + 4x = 7$

5. $7 - 4x = 5x^2$

Solving quadratic inequalities

To solve an inequality in x means to find the set of values of x which satisfy the inequality. Solving an inequality is very similar to solving an equation but there are, however, two important differences:

(i) the algebraic rules for inequalities are different.

(ii) an equation usually has a finite number of 'point' solutions; an inequality is usually satisfied by whole intervals greater than or less than a specific value.

Linear inequalities

Let us look at three linear inequalities just to refresh our memories about what it is that we are working with:

Example

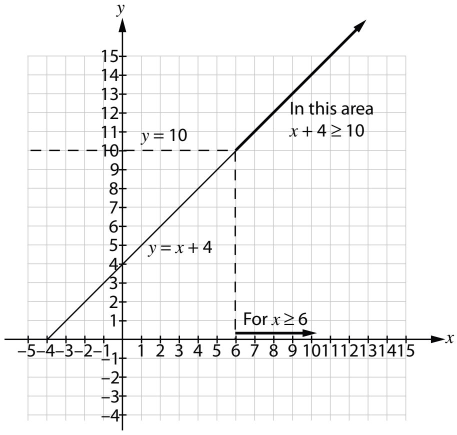
Example 1

Solve for x: $x + 4 \geq 10$

$x + 4 \geq 10$

$\therefore\ x \geq 10 - 4$

$\therefore\ x \geq 6$



Graphically:
We sketch the line $y = x + 4$. Now we will search for the x values for which the y values on the line are bigger than or equal to 10. To do this, we also sketch the line $y = 10$. At the point of their intersection, we find the solution for this inequality. It happens at the point where $x = 6$. But because we are looking for any value of x that will make $x + 4$ bigger than or equal to 10, we see that x must be bigger than or equal to 6.

Example

Example 2:

Solve for x in $3 \leq x + 4 \leq 8$

We want to find a solution of the form $a \leq x \leq b$

So we want x in the middle, and not $x + 4$.

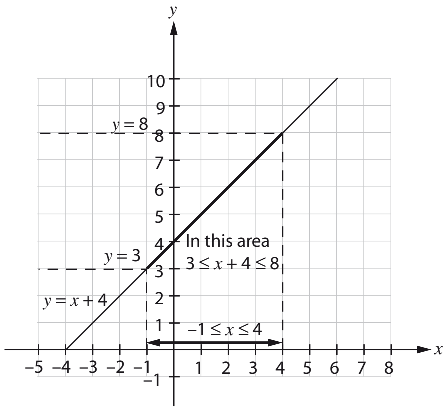
To obtain this we need to subtract 4 throughout.

$3 \leq x + 4 \leq 8$

$\therefore\ 3 - 4 \leq x + 4 - 4 \leq 8 - 4$

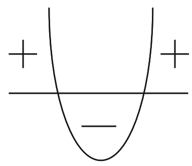
$\therefore\ -1 \leq x \leq 4$

So the solution set lies between negative one and four. Graphically the solution can be obtained as follows:



Here we need to consider three lines. They are $y = x + 4$, $y = 3$ and $y = 8$. Once we have sketched $y = x + 4$, we are searching where the y values on this line lie between 3 and 8. So we need to find the points of intersection where the lines $y = 3$ and $y = 8$ cross the line $y = x + 4$. We see that this happens at $x = -1$ and $x = 4$. Now since we want the set of values to lie between 3 and 8, we find our solution for x between -1 and 4.

Quadratic inequalities



In the examples to follow, the method for solving a quadratic inequality is vastly different from solving a linear inequality. Before we can solve any quadratic inequality, we must ensure that one side of the inequality is zero, by taking all terms to the LHS. After doing this, we ensure that all coefficients of x^2 are positive. Finally, we have to factorise this new inequaliy fully. Our solutions will be based on the diagram alongside. We can see that the y-values on the parabola are negative between its roots, and positive to either side of each root.

Example

Example 1: Solve for x in $x^2 - 3x \leq 4$

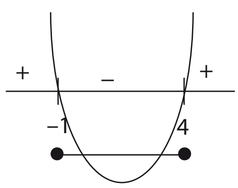
We ensure that the one side of this inequality is zero: $x^2 - 3x - 4 \leq 0$

Now we are ready to factorise:

$\therefore\ (x - 4)(x + 1) \leq 0$

> Since $x^2 - 3x - 4 \leq 0$ we are interested in the less than zero or 'negative' part of the parabola.

Now, in essence, this is a parabola (quadratic function) with two x-intercepts at $x = -1$ and at $x = 4$. So we can sketch the intercepts on a number line.



$-1 \leq x \leq 4$

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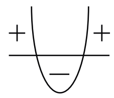
Our answer means that y is negative below the x-axis. So for all the x-values between the two roots, y will be negative on the parabola. To the left of -1 and the right of 4, the y-values are positive. So our solution: $-1 \le x \le 4$.



Example

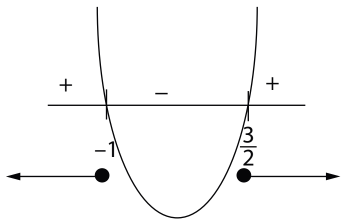
Example 2: Solve for x in $3 + x - 2x^2 \le 0$

We make sure that the coefficient of x^2 is positive:

$2x^2 - x - 3 \ge 0$. Now we can factorise this inequality:

$\therefore\ (2x - 3)(x + 1) \ge 0$ → This inequality sign tells us we are looking for the 'positive' part of the parabola.

The roots here are $x = -1$ and $x = \frac{3}{2}$



$$\therefore\ x \le -1 \text{ or } x \ge \frac{3}{2}$$

Example

Example 3: Solve for x in $x^2 + 2x + 6 \ge 0$

This inequality seems to have roots that are going to be difficult to find, since the factors of 6 will not add nor subtract to give the middle term 2. So we can complete the square to see what is happening here:

$$x^2 + 2x + 6 = x^2 + 2x + 1 + 5$$
$$= (x + 1)^2 + 5$$

Now if we consider the fact that the square is now contained in a bracket, then we can argue that $(x + 1)^2 \ge 0$ *for all x* since anything that is squared is always greater than or equal to zero (positive). Now if we add 5 to each side, it remains positive.

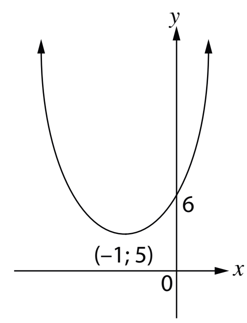
$\therefore\ (x + 1)^2 + 5 \ge 0 + 5$

$\therefore\ (x + 1)^2 + 5 \ge 5$ *for all x*

So $x^2 + 2x + 6 = (x + 1)^2 + 5$ is always positive for all values of x.

$\therefore$ x2 + 2x + 6 > 0 for all x $\in \mathbb{R}$

Graphically the situation is as follows:



Sometimes you can be required to show that a quadratic expression can only take on certain values, or possibly have a minimum or maximum value. Do not confuse these questions with solving an inequality. To find the extreme values, you need to contain x in a bracket. We use completing of the square to do so. This will help us to point out that the squared part is always positive, and then interpret the rest of the expression. Let's see how:

Example

Example 4: Show that $2x^2 - 4x + 5 > 0$

We need to **show that… not solve for x.** So we want to work at containing the x in a squared bracket:

$$
\begin{aligned}
2x^2 - 4x + 5 &= 2\left(x^2 - 2x + \tfrac{5}{2}\right) \\
&= 2\left(x^2 - 2x + 1 + \tfrac{3}{2}\right) \\
&= 2\left((x^2 - 2x + 1) + \tfrac{3}{2}\right) \\
&= 2\left((x + 1)^2 + \tfrac{3}{2}\right) \\
&= 2(x + 1)^2 + 3
\end{aligned}
$$

Now: $(x + 1)^2 \geq 0$ *for all x*

So: $2(x + 1)^2 \geq 0$ *for all x*

$\therefore\ 2(x + 1)^2 + 5 \geq 5$ *for all x*

$\therefore\ 2(x + 1)^2 + 5 > 0$

This answer shows that the minimum value of the expression $2x^2 - 4x + 5$ is indeed 5.

Example

Example 5: Show that $x^2 - 6x + 5 \geq -4$

Again we need not solve for x, but rather prove something about the expression. So we complete the square to get:

$$
\begin{aligned}
x^2 - 6x + 5 &= x^2 - 6x + 9 - 4 \\
&= (x - 3)^2 - 4
\end{aligned}
$$

Now: $(x - 3)^2 \geq 0$ *for all x*

So: $(x - 3)^2 - 4 \geq -4$ *for all x*

Finally this indicates that the minimum value of $x^2 - 6x + 5 = (x - 3)^2 - 4$ is negative four.

Example

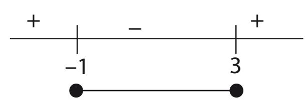
Example 6: For which values of x will $\sqrt{3 + 2x - x^2}$ be real.

We know that the square root of a negative number is non- real. So the number under the root must be positive for the root to be real.

$\therefore\ 3 + 2x - x^2 \geq 0$

$\therefore\ x^2 - 2x - 3 \leq 0$

$\therefore\ (x - 3)(x + 1) \leq 0$



$\therefore -1 \leq x \leq 3$

Activity

Activity 4

Solve for x in each of the following:

1. $x^2 \leq 9$

2. $4x - 2x^2 < 0$

3. $-7x^2 + 6x + 1 < 0$

4. $2x^2 - x - 1 \leq 0$

5. $x^2 + 2x + 1 \geq 0$

6. $(x^2 + 4)(x - 3) < 0$

7. $(x + 2)^2(x^2 + 4) > 0$

8. Show that: $-\frac{1}{2}x^2 + x + 4 \leq \frac{9}{2}$

9. Determine the minimum value of $\sqrt{x^2 - 8x + 25}$.

Solving simultaneous equations

In this section we are required to solve two variables, so we must be supplied with two equations. This is a very important section in your work, as you will need to apply it later to graphs and also analytical geometry. We will always work with one equation that is linear, and one which is quadratic.

- **In our application, we must always substitute the linear equation into the quadratic equation.**
- **Avoid, if possible, forming fractions when you work with the linear equation.**

When we solve simultaneous equations, we are finding the co-ordinates of the points where the graphs of the two curves cut one another.

METHOD:

1. **Isolate one of the variables in the linear equation.**
2. **Substitute into the second equation.**
3. **Solve for the unknown.**
4. **Substitute back into the equation from step 1.**
5. **Solve the for the other unknown.**

Example

Example 1

Solve for x and y in:

$x + 2y = 5$ and $2y^2 - xy - 4x^2 = 8$

Starting with the linear equation, we isolate one of the variables. Since $x + 2y = 5$, we will isolate the x since it has a coefficient of 1 and we will be avoiding forming unnecessary fractions.

So: $x = 5 - 2y \ldots (1)$ Now substitute into $2y^2 - xy - 4x^2 = 8$:

$2y^2 - (5 - 2y)y - 4(5 - 2y)^2 = 8$

$\therefore\ 2y^2 - 5y + 2y^2 - 4(25 - 20y + 4y^2) = 8$

$\therefore\ 4y^2 - 5y - 100 + 80y - 16y^2 - 8 = 0$

$\therefore\ 12y^2 - 75y + 108 = 0 \qquad (\div 3)$

$\therefore\ 4y^2 - 25y + 36 = 0$

$\therefore\ (4y - 9)(y - 4) = 0$

$\therefore\ y = \frac{9}{4}$ or $y = 4$

Now to solve for x since we have y, we substitute back into (1):

Then $x = 5 - 2\left(\frac{9}{4}\right)$ or $x = 5 - 2(4)$

$\therefore\ x = 5 - \frac{9}{2}$ or $x = 5 - 8$

$\therefore\ x = \frac{10 - 9}{2}$ or $x = -3$

$\therefore\ x = \frac{1}{2}$

So $x = \frac{1}{2}; y = \frac{9}{4}$ or $x = -3; y = 4$

We can write the answers as:

$(x; y) = \left(\frac{1}{2}\ ;\frac{9}{4}\right)$ *or* $(x; y) = (-3\ ; 4)$

Example 2

Solve for x and y in $y = -\frac{1}{2}x - 2$ and $y = x^2 - 2x - 3$

In both the equations, the y has already been isolated so we can just equate the two:

$-\frac{1}{2}x - 2 = x^2 - 2x - 3$

$\therefore -x - 4 = 2x^2 - 4x - 6$

$\therefore 2x^2 - 3x - 2 = 0$

$\therefore (x - 2)(2x + 1) = 0$

$\therefore x = 2$ or $x = -\frac{1}{2}$

If we now substitute these values back into the linear equation we get:

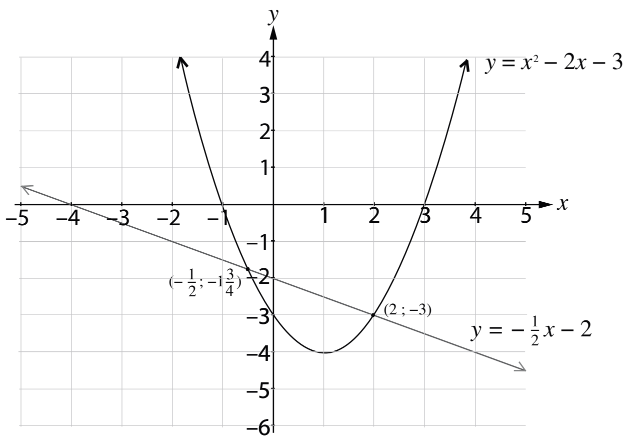
$y = -\frac{1}{2}(2) - 2$ or $y = -\frac{1}{2}\left(-\frac{1}{2}\right) - 2$

$\therefore$ y = – 1 – 2 or y = $\frac{1}{4} - 2$

$\therefore$ y = –3 or y = $-1\frac{3}{4}$

So: $(x; y) = (2; -3)$ *or* $(x; y) = \left(-\frac{1}{2}; -\frac{7}{4}\right)$

Graphically the situation will be as follows:



Activity 5

Solve for x and y in each of the following:

1. $x^2 - 2xy - 5y^2 = 3 = x - y$

2. $x + y = 9$ and $x^2 + xy + y^2 = 61$

3. $x - 2y = 3$ and $4x^2 - 5xy = 3 - 6y$

4. $(x - 1)^2 + (y - 2)^2 = 5$ and $2x + y + 1 = 0$

5. $(y - 3)(x^2 + 2) = 0$

Working with equations that contain fractions

Here it is important to know that you may get rid of the denominator provided that you state the restrictions on the variable x. Do not keep the denominators when you solve the equation.

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Example

Example 1

Solve for x in $\frac{2x}{2x-1} - \frac{10}{3} = \frac{1-2x}{2x}$

Starting the restrictions on x in the denominators:

$2x - 1 \neq 0$ *and* $2x \neq 0$

$\therefore x \neq \frac{1}{2}$ *and* $x \neq 0$.

Solving for x:

$3 \cdot 2x(2x) - 10 \cdot 2x(2x-1) = 3 \cdot (2x-1)(1-2x)$

$\therefore 12x^2 - 40x^2 + 20x = -3(2x-1)^2$

Multiply through by the LCD : $3 \cdot 2x(2x-1)$

$\therefore 12x^2 - 40x^2 + 20x = -12x^2 + 12x - 3$

$\therefore -16x^2 + 8x + 3 = 0$

$\therefore 16x^2 - 8x - 3 = 0$

$\therefore (4x-3)(4x+1) = 0$

$\therefore x = \frac{3}{4}$ or $x = -\frac{1}{4}$

Both these answers satisfy the restrictions and are therefore valid.

Example

Example 2

Solve for x in

$\frac{2x}{x-1} + \frac{3x}{1-x^2} = \frac{6}{x+1}$

We need to first factorise the denominator $(1 - x^2)$ completely. We can change the sign in the middle term so that

$1 - x^2 = -(x^2 - 1) = -(x-1)(x+1)$:

$\therefore \frac{2x}{x-1} - \frac{3x}{(x-1)(x+1)} = \frac{6}{x+1}$

Restr: $x \neq \pm 1$; LCD $(x-1)(x+1)$

$\therefore 2x(x+1) - 3x = 6(x-1)$

$\therefore 2x^2 + 2x - 3x = 6x - 6$

$\therefore 2x^2 - 7x + 6 = 0$

$\therefore (2x-3)(x-2) = 0$

$\therefore x = \frac{3}{2}$ or $x = 2$

Example

Example 3

Solve for x in

$\frac{x^2(x+1) - 2x(x+1) + (x+1)}{x^2 - 1} = 0$

Restr: $x^2 - 1 \neq 0$

$\therefore x^2 \neq 1 \rightarrow x \neq \pm 1$

$\therefore x^2(x+1) - 2x(x+1) + (x+1) = 0$ Factor out the common factor $x + 1$

$\therefore (x+1)(x^2 - 2x + 1) = 0$

$\therefore (x+1)(x-1)^2 = 0$

$\therefore x = -1$ or $x = 1$

But $x \neq \pm 1$

Therefore there is no solution to this problem!

Activity

Activity 6

Solve for x in each of the following:

1. $\frac{6}{x+1} - \frac{1}{x-1} = \frac{2}{x}$

2. $\frac{4x}{x^2-9} + \frac{3}{x-3} = \frac{2}{x+3} + 1$

3. $\frac{6}{x+1} - \frac{3x}{1-x^2} = \frac{2x}{x-1}$

4. $\frac{x}{x+1} - \frac{2}{1-x} = \frac{x^2+3}{x^2-1} + \frac{9}{4}$

Solutions to Activities

Activity 1

1. $x^2 - 7x + 10 = 0$
   $\therefore (x-2)(x-5) = 0$
   $\therefore x = 2$ or $x = 5$

2. $x^2 + 4x = 21$
   $\therefore x^2 + 4x - 21 = 0$
   $\therefore (x+7)(x-3) = 0$
   $\therefore x = -7$ or $x = 3$

3. $12x^2 - 6 = -17x - 12$
   $\therefore 12x^2 + 17x + 6 = 0$
   $\therefore (4x+3)(3x+2) = 0$
   $\therefore x = -\frac{3}{4}$ or $x = -\frac{2}{3}$

4. $x^2 = 6 - 5x$
   $\therefore x^2 + 5x - 6 = 0$
   $\therefore (x+6)(x-1) = 0$
   $\therefore x = -6$ or $x = 1$

5. $6x^2 + 11x + 3 = 0$
   $\therefore (2x+3)(3x+1) = 0$
   $\therefore x = -\frac{3}{2}$ or $x = -\frac{1}{3}$

6. $8x^2 - 14x + 3 = 0$
   $\therefore (4x-1)(2x-3) = 0$
   $\therefore x = \frac{1}{4}$ or $x = \frac{3}{2}$

7. $21x^2 = 41x - 10$

$\therefore 21x^2 - 41x + 10 = 0$

$\therefore (7x - 2)(3x - 5) = 0$

$\therefore x = \frac{2}{7}$ or $x = \frac{5}{3}$

8. $3x^2 + x = 14$

$\therefore 3x^2 + x - 14 = 0$

$\therefore (3x + 7)(x - 2) = 0$

$\therefore x = -\frac{7}{3}$ or $x = 2$

Activity 2

1. $2x - 5 = \frac{7}{x}$

$\therefore 2x^2 - 5x - 7 = 0$

$a = 2; b = -5; c = -7$

$\therefore x = \frac{5 \pm \sqrt{25 - 4(-14)}}{4}$

$\therefore x = \frac{5 \pm \sqrt{25 + 56}}{4}$

$\therefore x = \frac{5 \pm \sqrt{81}}{4}$

$\therefore x = \frac{5 \pm 9}{4}$

$\therefore x = \frac{7}{2}$ or $x = -1$

2. $\frac{1}{2}x^2 - 3x + 0{,}7 = 0$

$\therefore 5x^2 - 30x + 7 = 0$

$\therefore x = \frac{30 \pm \sqrt{900 - 4(35)}}{10}$

$\therefore x = \frac{30 \pm \sqrt{760}}{10}$

$\therefore x = \frac{30 \pm 2\sqrt{190}}{10}$

$x = \frac{15 \pm \sqrt{190}}{5}$

3. $4x^2 - 20x + 9 = 0$

$a = 4; b = -20; c = 9$

$\therefore x = \frac{20 \pm \sqrt{400 - 4(36)}}{8}$

$\therefore x = \frac{20 \pm \sqrt{400 - 144}}{8}$

$\therefore x = \frac{20 \pm \sqrt{256}}{8}$

$\therefore x = \frac{20 \pm 16}{8}$

$\therefore x = \frac{9}{2}$ or $x = \frac{1}{2}$

4. $2x^2 + 4x + 3 = 0$

$a = 2 ; b = 4 ; c = 3$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{16 - 4(6)}}{4}$

$= \frac{-4 \pm \sqrt{-8}}{4}$

x is non-real

Activity 3

1. $x^2 - 6x + 3 = 0$

$\therefore x^2 - 6x + 9 = -3 + 9$

$\therefore (x - 3)^2 = 6$

$\therefore x = 3 \pm \sqrt{6}$

$\therefore x = 5{,}45$ or $x = 0{,}55$

or

$x^2 - 6x + 3 = 0$

$\therefore 4x^2 - 24x + 36 = -12 + 36$

$\therefore (2x - 6)^2 = 24$

$\therefore 2x = 6 \pm 2\sqrt{6}$

$\therefore x = 3 \pm \sqrt{6}$

$\therefore x = 5{,}45$ or $x = 0{,}55$

2. $x^2 - 4x = -5$

$\therefore x^2 - 4x + 4 = -5 + 4$

$\therefore (x - 2)^2 = -1$

$\therefore x$ is non-real

or

$x^2 - 4x = -5$

$\therefore 4x^2 - 16x + 16 = -20 + 16$

$\therefore (2x - 4)^2 = -4$

$\therefore x$ is non-real

3. $2x^2 - x - 5 = 0$

$\therefore x^2 - \frac{x}{2} + \frac{1}{16} = \frac{5}{2} + \frac{1}{16}$

$\therefore \left(x - \frac{1}{4}\right)^2 = \frac{41}{16}$

$\therefore x = \frac{1 \pm \sqrt{41}}{4}$

$\therefore x = 1{,}85$ or $x = -1{,}35$

or

$2x^2 - x = 5$

$\therefore 16x^2 - 8x + 1 = 40 + 1$

$\therefore (4x - 1)^2 = 41$

$\therefore x = \frac{1 \pm \sqrt{41}}{4}$

$\therefore x = 1{,}85$ or $x = -1{,}35$

4. $3x^2 + 4x = 7$

$\therefore x^2 + \frac{4}{3}x + \frac{16}{36} = \frac{7}{3} + \frac{16}{36}$

$\therefore \left(x + \frac{4}{6}\right)^2 = \frac{100}{36}$

$\therefore x = \frac{-4 \pm 10}{6}$

$\therefore x = -\frac{7}{3}$ or $x = 1$

or

$3x^2 + 4x = 7$

$\therefore 36x^2 + 48x + 16 = 84 + 16$

$\therefore (6x + 4)^2 = 100$

$\therefore x = \frac{-4 \pm 10}{6}$

$\therefore x = -\frac{7}{3}$ or $x = 1$

5. $7 - 4x = 5x^2$

$\therefore 5x^2 + 4x = 7$

$\therefore x^2 + \frac{4}{5}x + \frac{4}{25} = \frac{7}{5} + \frac{4}{25}$

$\therefore \left(x + \frac{2}{5}\right)^2 = \frac{39}{25}$

$\therefore x = \frac{-2 \pm \sqrt{39}}{5}$

$\therefore x = 0{,}85$ or $x = -1{,}65$

or

$5x^2 + 4x = 7$

$\therefore 100x^2 + 80x + 16 = 140 + 16$

$\therefore (10x + 4)^2 = 156$

$\therefore x = \frac{-4 \pm 2\sqrt{39}}{10}$

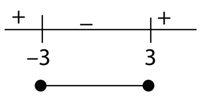
$\therefore x = \frac{-2 \pm \sqrt{39}}{5}$

$\therefore x = 0{,}85$ or $x = -1{,}65$

Activity 4

1. $x^2 \leq 9$

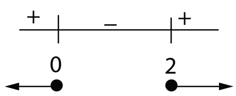
$\therefore (x - 3)(x + 3) \leq 0$



$\therefore -3 \leq x \leq 3$

2. $4x - 2x^2 < 0$

$\therefore 2x(x - 2) > 0$

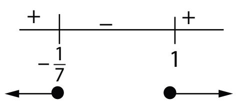


$\therefore x < 0 \text{ or } x > 2$

3. $-7x^2 + 6x + 1 < 0$

$\therefore 7x^2 - 6x - 1 > 0$

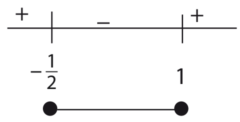
$\therefore (7x + 1)(x - 1) > 0$



$\therefore x < -\frac{1}{7} \text{ or } x > 1$

4. $2x^2 - x - 1 \leq 0$

$\therefore (2x + 1)(x - 1) \leq 0$



$\therefore -\frac{1}{2} \le x \le 1$

5. $x^2 + 2x + 1 \ge 0$

   $\therefore (x+1)^2 \ge 0$

   This is true for all x

6. $(x^2+4)(x-3) < 0$

   $Now: x^2 \ge 0 \rightarrow x^2 + 4 \ge 4$

   so for the result to be negative:

   $(x-3) < 0$

   $\therefore x < 3$

7. $(x+2)^2(x^2+4) > 0$

   $Now: x^2 + 4 \ge 4 > 0$

   So for the product to be positive:

   $(x+2)^2 > 0$

   $\therefore x \in R - \{-2\}$

8. $$\begin{aligned} -\frac{1}{2}x^2 + x + 4 &= -\frac{1}{2}(x^2 - 2x - 8) \\ &= -\frac{1}{2}((x^2 - 2x + 1) - 9) \\ &= -\frac{1}{2}((x-1)^2 - 9) \\ &= -\frac{1}{2}(x-1)^2 + \frac{9}{2} \end{aligned}$$

   So the maximum is $\frac{9}{2}$

9. $\sqrt{x^2 - 8x + 25} = \sqrt{x^2 - 8x + 16 + 9}$

   $= \sqrt{(x-4)^2 + 9}$

   The minimum will happen when $x = 4$: i.e. $\sqrt{9} = 3$.

Activity 5

1. $x^2 - 2xy - 5y^2 = 3 \dots (1)$

   $x = 3 + y \dots (2)$

   $(1) \rightarrow (2): (3+y)^2 - 2y(3+y) - 5y^2 = 3$

   $\therefore y^2 + 6y + 9 - 6y - 2y^2 - 5y^2 - 3 = 0$

   $\therefore 6y^2 - 6 = 0$

   $\therefore y^2 = 1$

   $\therefore y = \pm 1$

   Now: $x = 3 \pm 1$

   $\therefore x = 4 \ or\ x = 2$

   So $(x; y) = (4; 1)$ or $(x; y) = (2; -1)$

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2. $x = 9 - y$ ...(1) and $x^2 + xy + y^2 = 61$ …(2)

$(1) \rightarrow (2) : (9 - y)^2 + (9 - y)y + y^2 = 61$

$\therefore\ y^2 - 18y + 81 + 9y - y^2 + y^2 - 61 = 0$

$\therefore\ y^2 - 9y + 20 = 0$

$\therefore\ (y - 5)(y - 4) = 0$

$\therefore\ y = 5$ or $y = 4$

So: $x = 9 - 5$ or $x = 9 - 4$

$\therefore\ x = 4$ or $x = 5$

$\therefore\ (x; y) = (4; 5)$ or $(x; y) = (5; 4)$

3. $x = 2y + 3$ and $4x^2 - 5xy = 3 - 6y$

$\therefore\ 4(2y + 3)^2 - 5y(2y + 3) = 3 - 6y$

$\therefore\ 4(4y^2 + 12y + 9) - 10y^2 - 15y + 6y - 3 = 0$

$\therefore\ 16y^2 + 48y + 36 - 10y^2 - 9y - 3 = 0$

$\therefore\ 6y^2 + 39y + 33 = 0$ $(\div 3)$

$\therefore\ 2y^2 + 13y + 11 = 0$

$\therefore\ (2y + 11)(y + 1) = 0$

$\therefore\ y = -\frac{11}{2}$ or $y = -1$

So: $x = 2\left(-\frac{11}{2}\right) + 3$ or $x = 2(-1) + 3$

$\therefore\ x = -8$ or $x = 1$

$\therefore\ (x; y) = \left(-8; -\frac{11}{2}\right)$ or $(x; y) = (1; -1)$

4. $(x - 1)^2 + (y - 2)^2 = 5$ and $y = -2x - 1$

$\therefore\ (x - 1)^2 + (-2x - 1 - 2)^2 = 5$

$\therefore\ x^2 - 2x + 1 + 4x^2 + 12x + 9 - 5 = 0$

$\therefore\ 5x^2 + 10x + 5 = 0$ $(\div 5)$

$\therefore\ x^2 + 2x + 1 = 0$

$\therefore\ (x + 1)^2 = 0$

$\therefore\ x = -1$

So: $y = -2(-1) - 1 = 1$

$\therefore\ (x; y) = (-1; 1)$

5. $(y - 3)(x^2 + 2) = 0$

$\therefore\ y - 3 = 0$ or $x^2 + 2 = 0$

$\therefore\ y = 3$ only since $x^2 = -2$ has no real solution.

Activity 6

1. $\frac{6}{x + 1} - \frac{1}{x - 1} = \frac{2}{x}$

LCD: $x(x - 1)(x + 1)$

Restrictions: $x \neq 0$ *and* $x \neq \pm 1$

$\therefore\ 6x(x - 1) - x(x + 1) = 2(x^2 - 1)$

$\therefore\ 6x^2 - 6x - x^2 - x = 2x^2 - 2$

$\therefore\ 3x^2 - 7x + 2 = 0$

$\therefore\ (3x - 1)(x - 2) = 0$

$\therefore\ x = \frac{1}{3}$ or $x = 2$

2. $\frac{4x}{(x-3)(x+3)} + \frac{3}{x-3} = \frac{2}{x+3} + 1$

LCD: $(x - 3)(x + 3)$

Restrictions: $x \neq \pm 3$

$\therefore\ 4x + 3(x + 3) = 2(x - 3) + (x - 3)(x + 3)$

$\therefore\ 4x + 3x + 9 = 2x - 6 + x^2 - 9$

$\therefore\ x^2 - 5x - 24 = 0$

$\therefore\ (x - 8)(x + 3) = 0$

$\therefore\ x = 8$ or $x = -3\ n.a.$

3. $\frac{6}{x+1} + \frac{3x}{(x-1)(x+1)} = \frac{2x}{x-1}$ Note from earlier: $-(1 - x^2) = (x - 1)(x + 1)$

LCD: $(x - 1)(x + 1)$

Restriction: $x \neq \pm 1$

$\therefore\ 6(x - 1) + 3x = 2x(x + 1)$

$\therefore\ 6x - 6 + 3x = 2x^2 + 2x$

$\therefore\ 2x^2 - 7x + 6 = 0$

$\therefore\ (2x - 1)(x - 3) = 0$

$\therefore\ x = \frac{1}{2}$ or $x = 3$

4. $\frac{x}{x+1} + \frac{2}{x-1} = \frac{x^2+3}{(x-1)(x+1)} + \frac{9}{4}$ $-(1 - x) = (x - 1)$

LCD: $4(x - 1)(x + 1)$

Restriction: $x \neq \pm 1$

$\therefore\ 4x(x - 1) + 2 \cdot 4(x + 1) = 4(x^2 + 3) + 9(x^2 - 1)$

$\therefore\ 4x^2 - 4x + 8x + 8 = 4x^2 + 12 + 9x^2 - 9$

$\therefore\ 9x^2 - 4x - 5 = 0$

$\therefore\ (9x + 5)(x - 1) = 0$

$\therefore\ x = -\frac{5}{9}$ or $x = 1\ n.a.$