LINEAR PROGRAMMING

Traditionally we apply our knowledge of Linear Programming to help us solve real world problems (which is referred to as **modelling**). Linear Programming is often linked to the field of economics since we usually wish to maximise or minimise a numerical function\equation subject to certain conditions, known as **constraints**. Basically we wish to determine the best way to do something. For example, we can obtain the best profit based on given conditions.

Now this may all sound a bit confusing, but it isn't really, and we will return to this section again in Grade 12. This is, however, a section where your ability to read and extract information will be critical. Lots of new terminology will be added to your mathematics vocabulary so ensure that you learn these new words so that you always understand what is being asked.

The first thing that we need to do is ensure that we are familiar with some of the work already covered in grade 9 and 10. Since this section relies heavily on our ability to draw and interpret **straight line graphs**, we will start by revising this particular topic.

We must also be able to solve **linear equations** (find the points of intersection of two graphs) as well as **linear inequalities**. So let's begin our revision.

Straight line graphs

(Revision of Grade 10)

Although you are probably familiar with the straight line graph being written in the form $y = mx + c$, it can, and more than likely will, be written in the form $ax + by = c$.

So, how do we draw a graph of the form $ax + by = c$?

Probably the easiest way is by using the dual intercept method. This basically means that we must find the x- and y-intercepts and then draw the straight line through these points.

Finding the Intercepts:

e.g. Sketch the graph of $2x - 3y = 6$

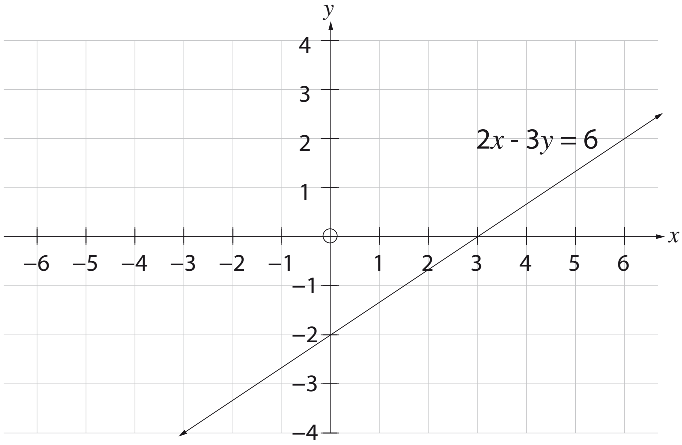
x-intercept: Substitute $y = 0$ (and solve for x) $\therefore x = 3$

point (3 ; 0)

y-intercept: Substitute $x = 0$ (and solve for y) $\therefore y = -2$

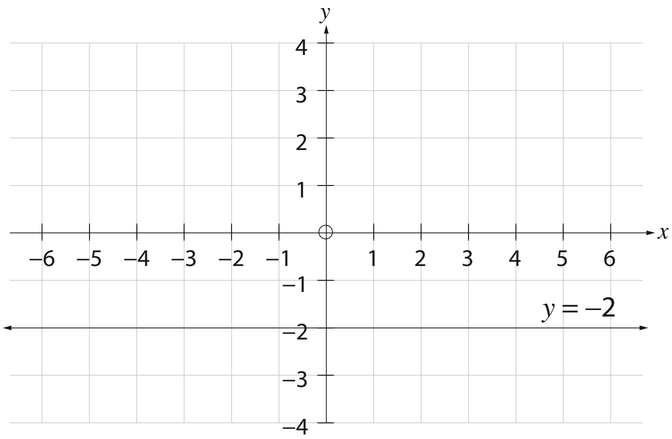
point (0 ; –2)

Now, we plot these points, draw a straight line through them, and we have our sketch.



Graphs of the form: $y = c$

e.g. Sketch $y = -2$



This is the horizontal line which has all coordinates of the form $(x\,;-2)$.

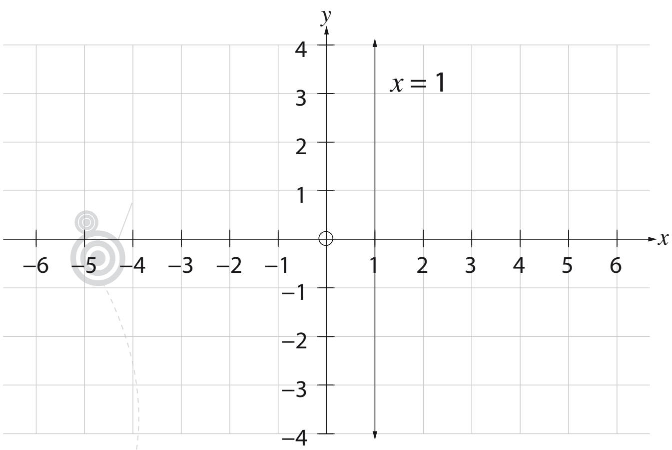
This line will always cut the y-axis at -2.

Graphs of the form: $x = c$

e.g. Sketch $x + 3 = 4$ Equations will not always be written in the simplest form.

simplifying $x = 4 - 3$

$x = 1$

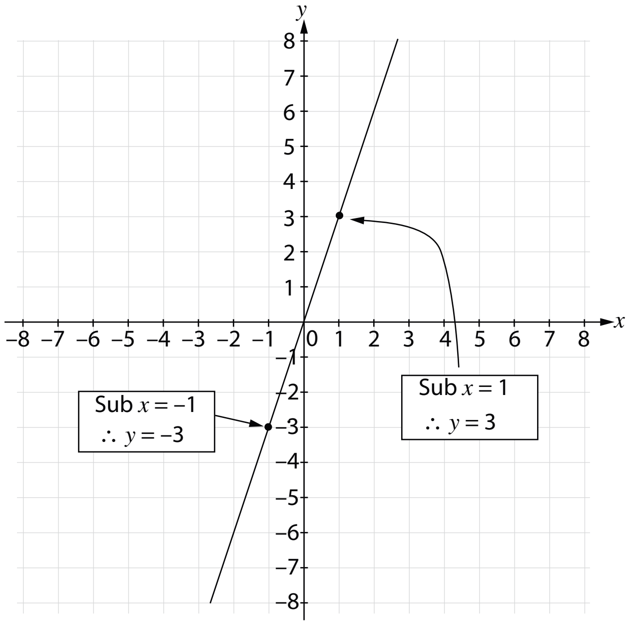


This is the vertical line which has all coordinates of the form $(1;\,y)$.

This line will always cut the x-axis at 1.

Graphs of the form $y = mx$

e.g. Sketch $y = 3x$



Notice that this graph is of the form: $y = mx + c$, but $c = 0$, $\therefore$ **this graph always passes through the origin**. So, we cannot draw the graph using the dual intercept method since there is only 1 intercept.

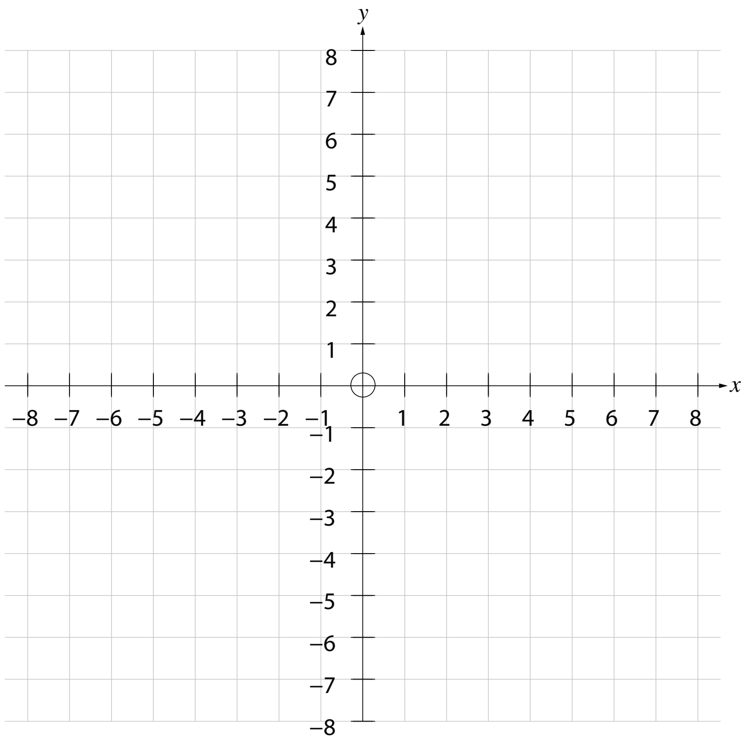
To draw this graph all we need to do is find 1 other point, since we already have the point (0; 0).

To find another point we must choose a suitable x-value, and then find the corresponding y-value.

Activity 1

Activity

Sketch each of the graphs on the same set of axes, which has been provided for you.



Where appropriate, determine the intercepts first (i.e. draw the graphs using the dual-intercept method).

Label each graph carefully.

1. $y + 4x = 8$
2. $2x - 3y = 12$
3. $x = 4$

4. $y = \frac{1}{2}x$

5. $y = 2$

One of the important skills required in this section is to find the points of intersection of two graphs. So, for example, let's find the points of intersection of the graphs $y = 2x + 4$ and $2y - 6x = 6$.

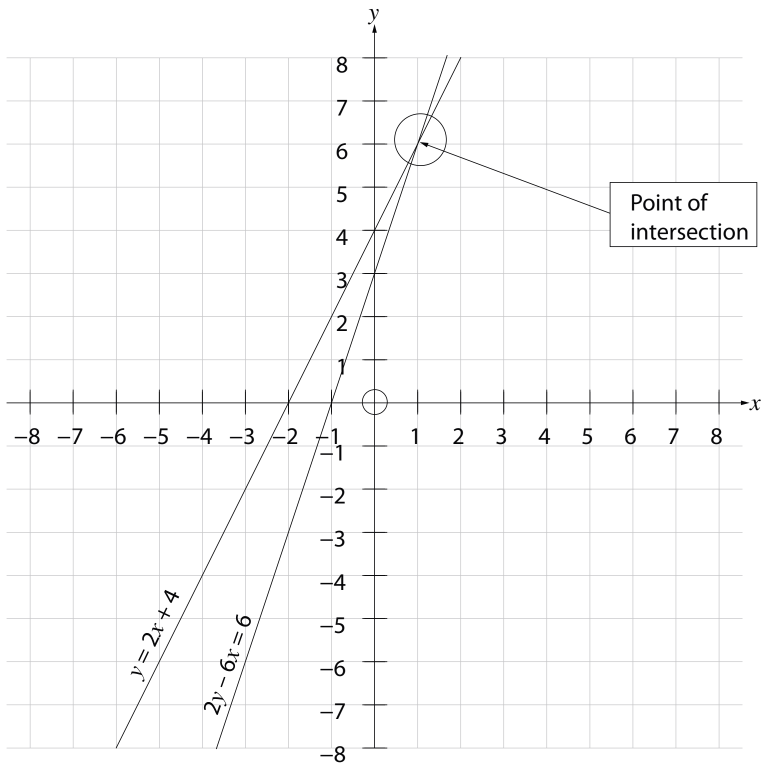
Graphically

One way to find the intersection of two graphs is to draw them and see where they cut each other. (Just remember that parallel lines have the same gradient so it is quite possible that they never intersect, or they always intersect/overlap.)

Drawing the graphs, using the dual intercept method:

From $y = 2x + 4$ y-intercept: (0 ; 4) x-intercept (–2 ; 0)

From $2y - 6x = 6$ y-intercept: (0 ; 3) x-intercept (–1 ; 0)



From the graphs it is quite clear that the point of intersection occurs at the point (1 ; 6). However it is not always possible, or always accurate, to determine the solution graphically. This is why we find the point of intersection algebraically.

Algebraically

Remember that finding the intersection of two graphs means that we must solve simultaneously. This was taught to you in Grade 10, but let's do a very quick recap.

$y = 2x + 4$ … (1)

$y - 3x = 3$ … (2)

Step 1: Choose the easier of the two equations, in order to isolate one of the variables.

From (1) $\quad y = 2x + 4$

Step 2: Substitute, in place of this variable, into the other equation ($y - 3x = 3$).

Sub into (2) $\quad (2x + 4) - 3x = 3$

Step 3: Solve this equation for the single unknown.

$2x + 4 - 3x = 3$

$-x = -1$

$\therefore x = 1$

Step 4: Determine the value of the other unknown (i.e. substitute into either of the equations).

Sub $x = 1$ into (1) $\quad y = 2(1) + 4$

$y = 6$

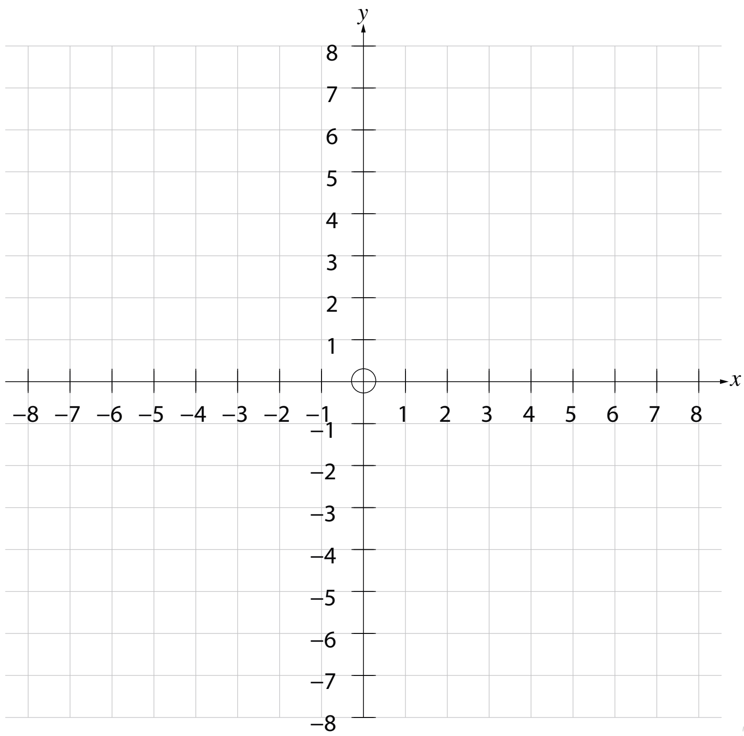
So the point of intersection is (1 ; 6), which is consistent with the graphical solution.

Activity 2

Determine the points of intersection for each of the given straight line graphs.

i) Determine first, the point of intersection algebraically.

ii) Then draw the graphs and verify the point of intersection graphically. Use the set of axes provided.



1. $2x - y = 8 \,;\, x + y = 4$
2. $x = 3;\ 4y - 2x = 10$

Linear inequalities

In Grade 10 you solved linear inequalities, which are very similar to linear equations.

e.g. Solve $2y - 1 < y + 6$

$2y - y < 6 + 1$

$y < 7$

Just like we would solve a linear equation, we take all the variables to the 'same side', the numbers to the 'other side', and then solve from there.

Now we ask ourselves, "What does this solution represent, and how does it differ from an equation?"

Well, first of all, let's compare it to the 'same' equation ($2y - 1 = y + 6$) which gives the solution $y = 7$. This linear equation has one solution only.

However this inequality has infinitely many solutions that satisfy the condition $y < 7$. An inequality has a **set of values** that form the solution set.

Graphing linear inequalities

What would the graph of $2x - 3y \geq 6$ look like?

We change an equation into an inequality when we change the = sign to either > or <. Clearly this has an effect on how we draw the inequality. We are dealing with values that are either 'bigger than' or 'smaller than' a particular value.

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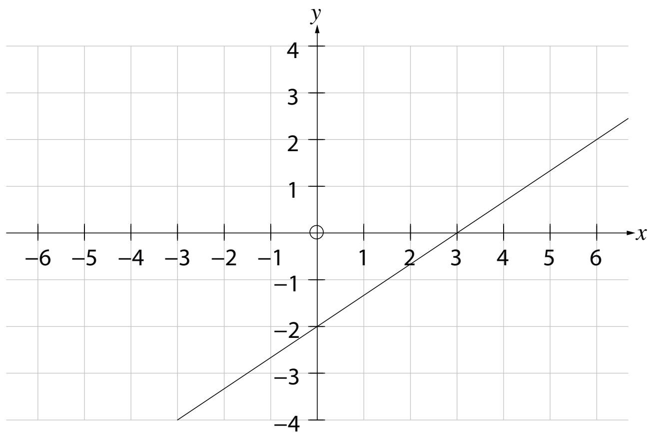
You may have noticed that we encountered this 'equation'($2x - 3y = 6$) earlier in the section. But how is this graph changed when we change the equation to read $2x - 3y \geq 6$?



One of two scenarios occur. Since we know that an inequality results in a set of values that satisfy a condition, our full solution lies either 'above' or 'below' the equation line. And the way that we represent this is by shading above or below. Both options are illustrated, although only one of the figures represents the correct solution set. Can you determine which figure is correct?

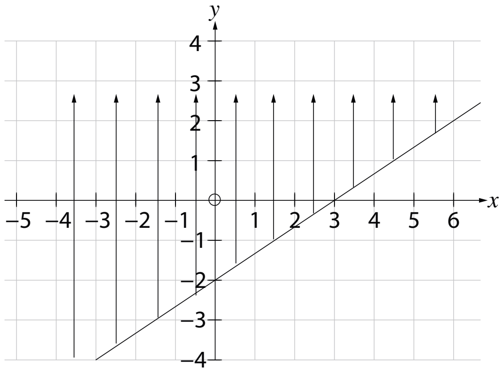


Figure 1

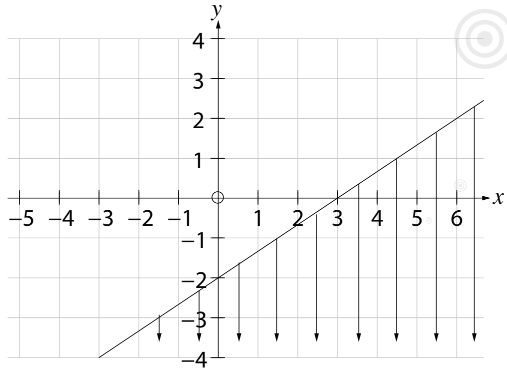
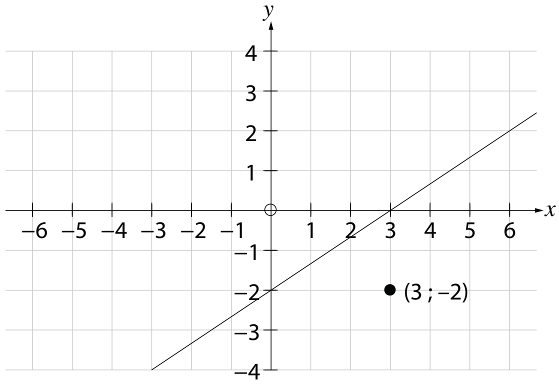


Figure 2

One way to determine which side to shade is to choose a 'test' point. Pick any point on the Cartesian plane. Lets choose the point (3 ; –2). (You may choose any point that is clearly above or below the line.)



Now substitute this point (3 ; –2) into the inequality $2x - 3y \geq 6$.

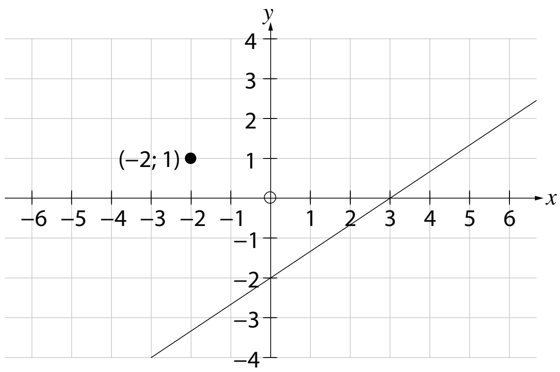
$2(3) - 3(-2) \geq 6$

$6 + 6 \geq 6$

$12 \geq 6$ …which is of course true.

So this means we must shade below since this is where the 'TRUE' solution is. So Figure two is correct.

But what would have happened if we had chosen the point, say, (–2 ; 1) initially?



Again, we would substitute this point into the original inequality $2x - 3y \geq 6$ to get:

$2(-2) - 3(1) \geq 6$

$-4 - 3 \geq 6$

$-7 \geq 6$ …which is of false, meaning that we must shade on the other side of the line, i.e. we must shade below.

We always shade where the inequality is TRUE.

Lets recap how we sketch a linear inequality:

1) Start by drawing the standard linear equation (i.e. replace $>$ or $<$ by $=$).

2) Pick a 'test' point and substitute these values in place of x and y in the original inequality. If the test point is 'true' then shade on the test point side of the line, otherwise shade on the other side.

When is the sketch of a linear inequality a solid line and when is it a dotted line? (Irrespective of the shading.)

If our inequality is $>$ or $<$ then the line is dotted.

If our inequality is $\geq$ or $\leq$ then the line is solid.

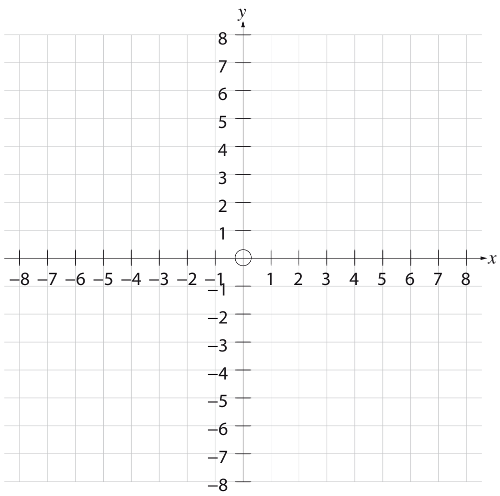
Activity

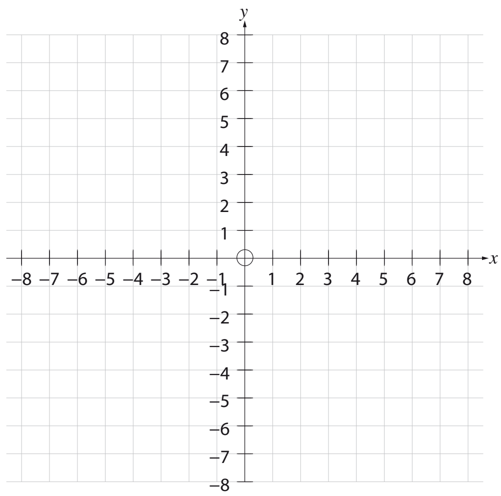
Activity 3

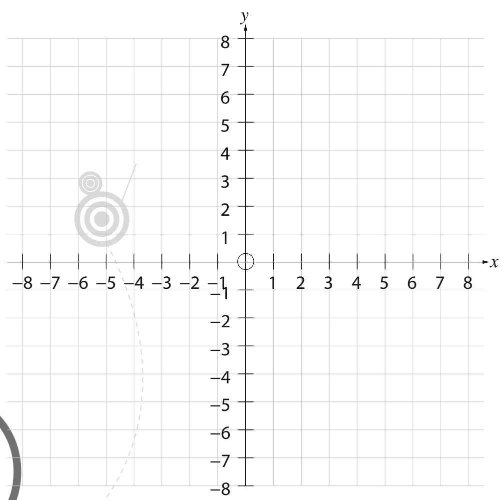
Sketch each of the following inequalities. Use a separate set of axes for each graph.

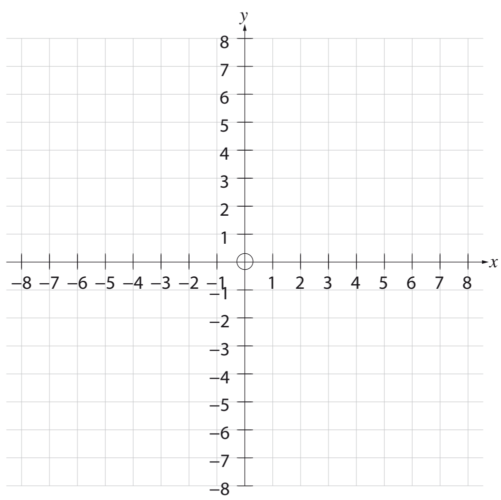
1. $x + y \leq 4$
2. $4y - 2x \geq 10$
3. $4x - 2y \geq 12$
4. $x \geq 6$

Use the set of axes provided.









In due course you will learn to *shade* by inspection.

If you rearrange the inequality to make y the subject of the formula, i.e. $y < \ldots$ or $y > \ldots$ then it is relatively easy to see if you must shade above or below the line.

if $y < \ldots$	SHADE BELOW
if $y > \ldots$	SHADE ABOVE

So, for our earlier example, $2x - 3y \geq 6$, where we used a test point to conclude that our shading was below the equation line , we could have simply done the following:

$2x - 3y \geq 6$	Rearrange the equation, making y the 'subject' of
$2x - 6 \geq 3y$	the formula.
$\frac{2x}{3} - \frac{6}{3} \geq \frac{3y}{3}$	
$\frac{2x}{3} - 2 \geq y$	
$\therefore\ y \leq \frac{2x}{3} - 2$	Which means we shade *below*.

The feasible region

We are now in a position where we can start exploring the basics of linear programming. This is where we will encounter **constraints**, which will enable us to create the **feasible region**.

In the 'real world', like in industry/business, production/manufacture is usually constrained by certain factors. Like the number of hours in a day, or the amount of raw materials that we have to manufacture something, or the budget that we are working to, and so the list goes on. Obviously we want to represent these constraints (practical restrictions) in the form of mathematical inequalities, so that we can represent these conditions graphically.

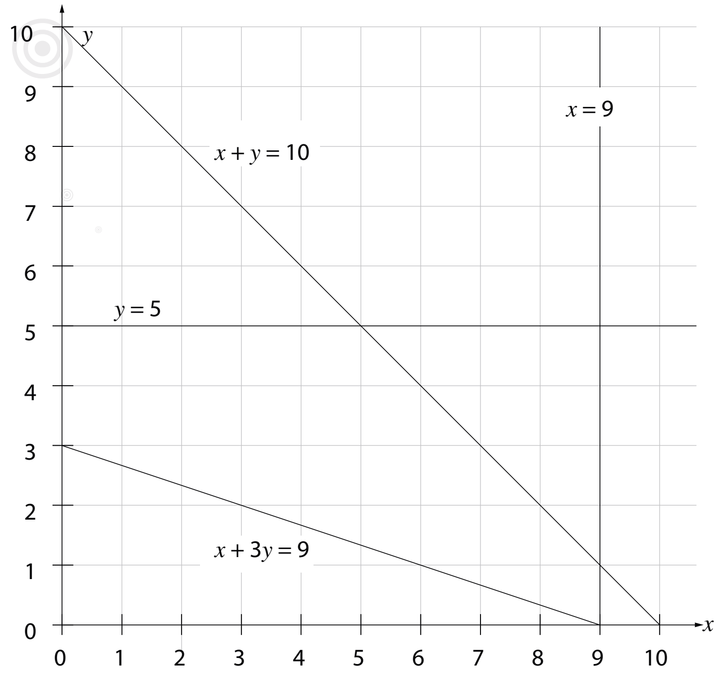
When we have all of the constraints, and we have created inequalities for each of them, we can sketch the resultant set of inequalities on the same set of axes. Remember that because we are sketching inequalities we end up with shading above or below the graph. When all of these graphs have been sketched there will be lots of shading! However we are only interested in the region where all of the shading overlaps. Do you know why?

This is the region where all of the conditions have been met, which is called the feasible region. Let's illustrate this concept.

On the same set of axes sketch the following inequalities:

$$\left.\begin{array}{l} y \leq 5 \\ x \leq 9 \\ x + y \leq 10 \\ x + 3y \geq 9 \end{array}\right\}$$

> Remember to sketch each straight line graph first, using the dual intercept method, and only then apply the shading. For $y < \ldots$ shade below and for $y > \ldots$ shade above. Similarly, for $x < \ldots$ shade to the left and for $x > \ldots$ shade to the right.



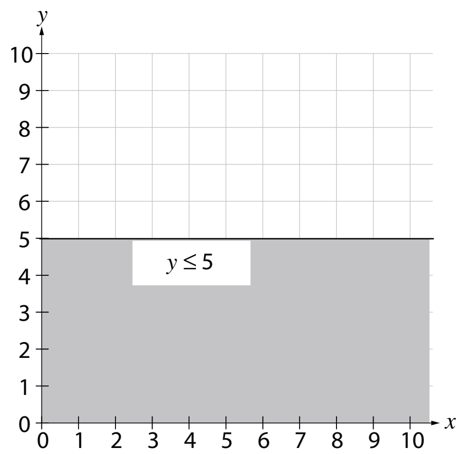
Here the straight line graphs have been sketched without any shading yet. (So we have not graphed the 'inequalities' yet.

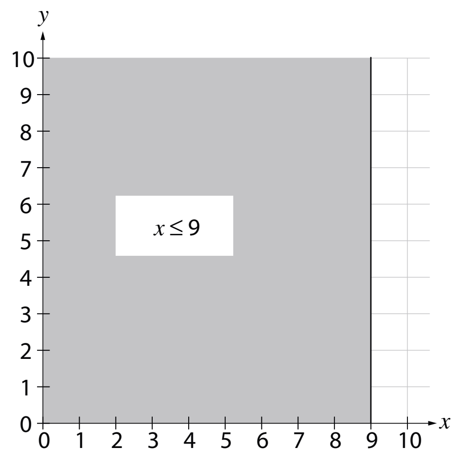
Let's now investigate the shading for each inequality individually, and only then combine them all together.

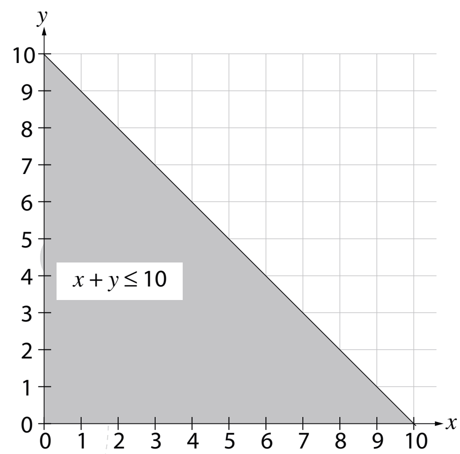
All the correct shading has now been applied (see below).

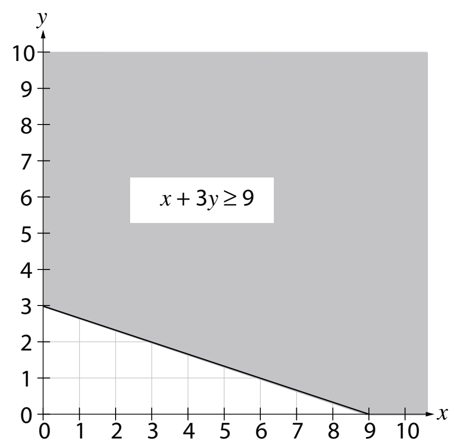
Now try to visualise the region where **ALL** of the shading will overlap from the 4 sketched inequalities.

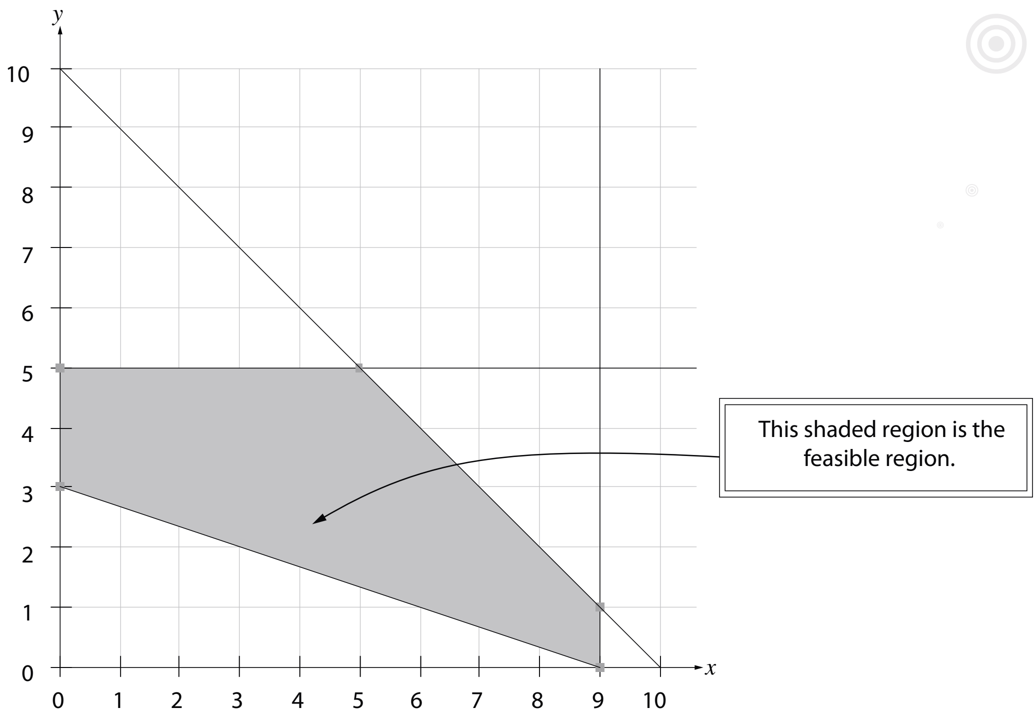
This is our feasible region.











Activity 4

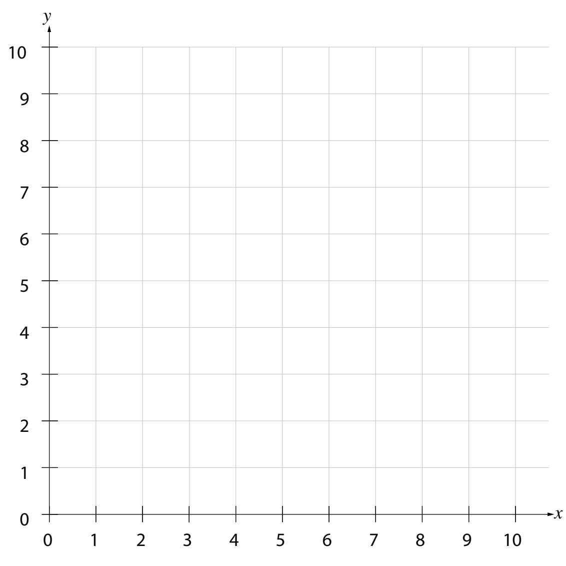
Activity

Determine the feasible region, based on the given constraints, for each of the following:

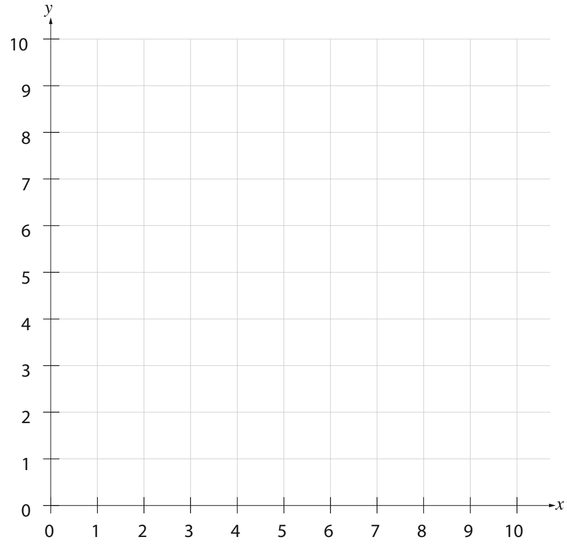
Represent your solution in the form of a sketch. Use a separate set of axes for each answer.

1. $y \leq 6$; $2x + y \leq 10$; $2x + 3y \geq 12$

Use the set of axes provided.



2. $x > 2$; $y \geq 2$; $3x + 6y \geq 30$



Lets look at a real world problem and see how we can create the feasible region to help us optimise our solution.

Example

You are the owner of a company called *Ipods-4-Us*, which manufacture two different types of Ipods. The first type, called *Video Junkie*, are more expensive and have superior graphics to the second type, called *Music Box*, which are cheaper and can only be used to play music.

Based on certain production factors at your factory, a maximum of 140 *Video Junkies*, and a maximum of 210 *Music Boxes* can be manufactured per week. Collectively no more than 260 Ipods of either type can be manufactured per week.

> Most questions, like this one, will often contain additional, unnecessary information. You must learn to **sift out the useful information** which will enable you to form your constraint equations.

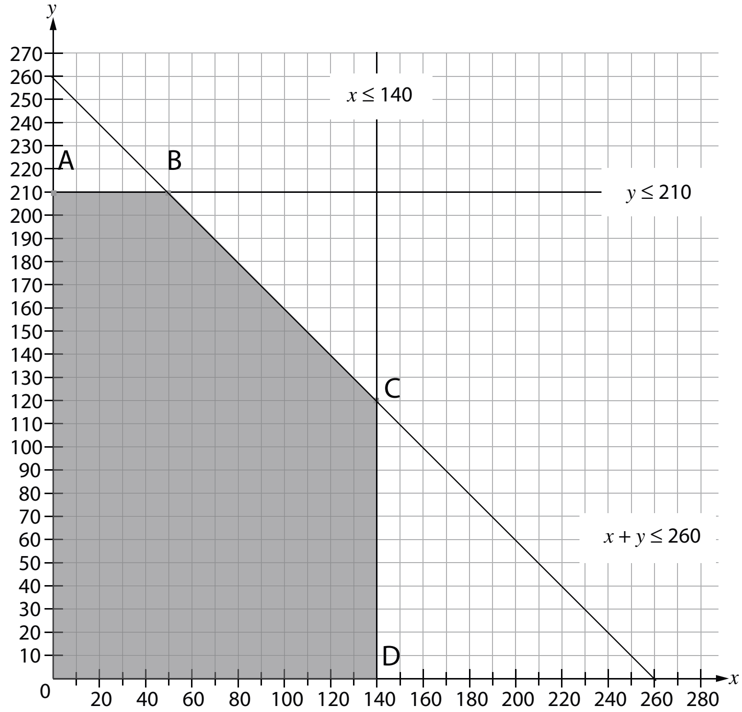
Let the number of *Video Junkie* Ipods produced be x and the number of *Music Box* Ipods be y.

Since we are going to eventually graph the information on the Cartesian plane, we always express the number of each type of product in terms of x and y.

1. Determine the constraints for the given information.
2. Represent the constraints on a set of axes. Clearly indicate the feasible region.
3. It is further given that you make a profit of R220 per *Video Junkie* and R150 per *Music Box* manufactured. Determine how many of each Ipod should be manufactured in order to maximise your profits.

Solution

1. $x \leq 140$; $y \leq 210$; $x + y \leq 260$
2. Now we graph the inequalities on the same set of axes, thus enabling us to determine the feasible region. This is illustrated below.



Note: The questions that usually follow, involving the calculation of the optimal solution, are based on the feasible region since this region represents all the possible 'combinations' that satisfy all the necessary conditions.

This general equation that we form (in this case, the profit equation) to find the optimal solution is called the **objective function**.

3. For this question we must start by forming a profit function.

$P = 220x + 150y$ ← Objective Function

profit of R220 per *Video Junkie* … $220x$

profit of R150 per *Music Box* … $150y$

Now in order to maximise the profit we must choose the biggest x and y (maximum number of *Video Junkie* and *Music Box* Ipods) values that are within the feasible region. The biggest values will lie along the boundary of the feasible region, at the points of intersection of the sketched graphs, which are the points A, B, C and D. Only one of these points will give us the maximum profit and our goal is to determine which is the correct point – optimal combination.

Note: If it is not clear what the actual coordinates of the points of intersection are, we can solve simultaneously. So, for example, point C is the point where the line $x = 140$ and $x + y = 260$ intersect, giving the coordinates (140; 120).

It should be clear that only the point B(50; 210) or C(140; 120) can yield the optimal combination, so only these two points need to be considered. By substituting the respective x and y values into the objective function we can check to see which point gives the maximum profit.

B: P = 220(50) + 150(210) = 42 500 i.e. 50 *Video Junkies* and 210 *Music Boxes*

C: P = 220(140) + 150(120) = 48 800 i.e. 140 *Video Junkies* and 120 *Music Boxes*

Since this is the greater profit, this means that this combination is the optimal combination.

Activity

Activity 5

For each of the following, write only the set of inequalities that satisfy the given constraints:

1. You are the owner of a swimming pool company that installs 2 types of fibreglass pools. The first is called the **Basic** and the second, a much fancier and more expensive swimming pool is called the **Exclusive**. Suppose that your firm produces x units of the Basic and y units of the Exclusive per month. It takes $1\frac{1}{2}$ days to install each Basic and 2 days to install each Exclusive.

   Your staff are at work for 24 days per month

   You cannot house more than 14 pools in your warehouse and your stock only arrives once a month, which means that you cannot install more than 14 pools per month

   At least 4 Exclusive pools must be installed per month

   The number of Exclusive pools that you install must be at least half the numberof Basic

   Two of the constrait inequalities are $x \geq 0$ and $x \geq 0$ where $x,y \in \mathbb{N}$.

   Write down the other 4 inequalities.

2. You manufacture cane chairs which you sell at a local market every Sunday. You manufacture a single-seater and a 2-seater couch. Suppose you can produce x units of the single-seater and x units of the 2-seater couch per **week**. You work 6 days a week (excluding Sunday, when you sell the chairs at the market), and up to 16 hours per day. Manufacturing a chair involves a two stage process; the basic assembly, and then the painting of the chair. It takes 4 hours to manufacture, and then 3 hours to paint the single-seater, and 7 hours to manufacture and 4 hours to varnish the 2-seater couch.

   At least two single-seaters must be manufactured for each 2-seater couch.

   A minimum of 10 single-seater couches must be manufactured.

   Due to the limited space that you have at the market, you cannot produce more than 6 2-seater couches.

   Two of the constraint inequalities are $x \geq 0$ and $y \geq 0$ where $x,y \in \mathbb{N}$.

   Write down the other 4 inequalities.

Activity 6

A retailer wishes to buy a maximum of 40 computers. He can buy either type A for R3 000 each or type B for R6 000 each. R180 000 has been budgeted for the purchase of the computers. However, at least 10 of **each,** type A and type B, must be purchased and at most 30 of type A.

Let x represent the number of type A and y represent the number of type B.

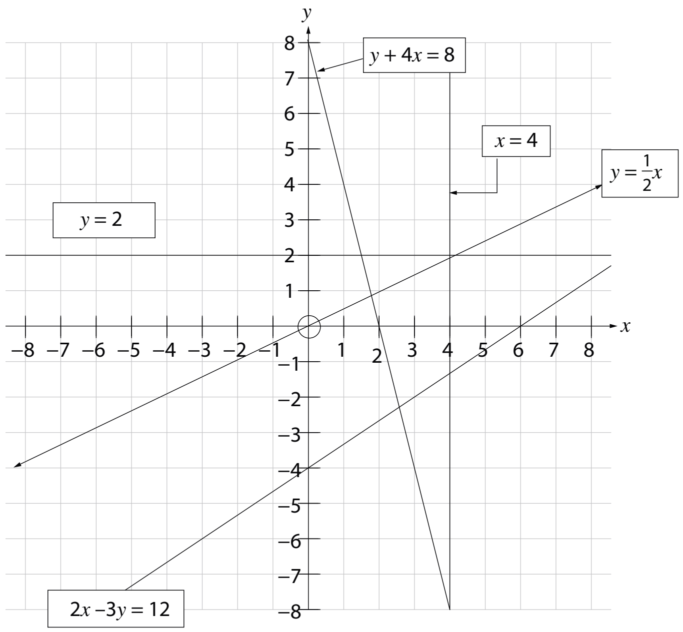
1. Write down the constraint inequalities.
2. Represent the constraints graphically on the set of axes provided and shade the feasible region.
3. The retailer makes a profit of R600 on each type A computer and R1 000 on each type B computer. If he sells all his stock, write an equation for the profit function, P.
4. Determine the maximum profit.

That brings us to the end of the Grade 11 modelling section, which really serves as an introduction to the Grade 12 section on modelling.

Solutions to Activities

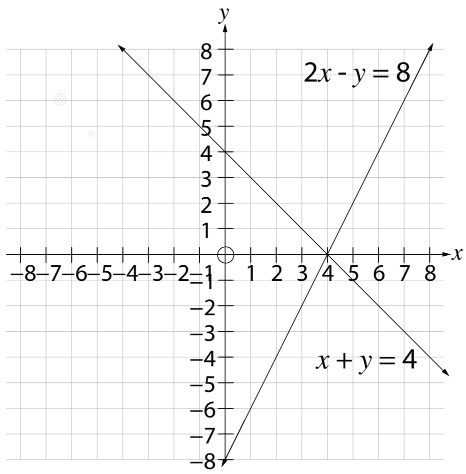
Activity 1

1.	$y + 4x = 8$	x-int: 2	y-int: 8
2.	$2x - 3y = 12$	x-int: 6	y-int: –4
3.	$x = 4$	vertical line through $x = 4$	
4.	$y = \frac{1}{2}x$	through origin and (2;1) (or any other correct point)	
5.	$y = 2$	horizontal line through $y = 2$	



Activity 2

1. $2x - y = 8 \ldots(1)\ ;\ x + y = 4 \ldots(2)$



Solving simultaneously:

From (2):

$y = 4 - x$ Sub into equation (1)

$2x - (4 - x) = 8$

$2x - 4 + x = 8$

$3x = 12$

$x = 4$

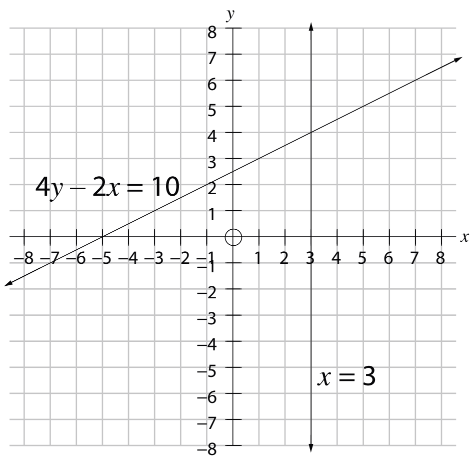
Substitute $x = 4$ back into $y = 4 - x$

$y = 4 - (4)$

$y = 0$

This point of intersection, (4; 0), is also illustrated graphically alongside.

2. $x = 3 \ldots (1);\ 4y - 2x = 10 \ldots (2)$



Solving simultaneously:

Substitute (1) into equation (2):

$4y - 2(3) = 10$

$4y - 6 = 10$

$4y = 16$

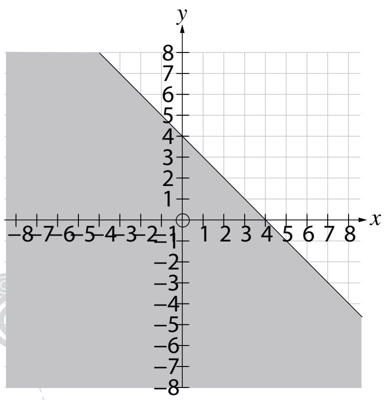
$y = 4$

And $x = 3$.

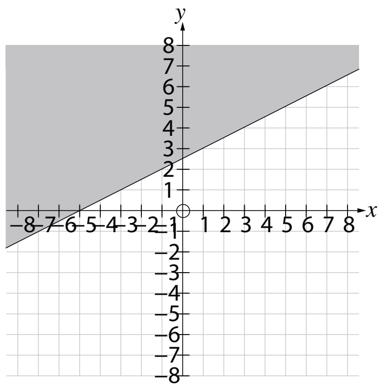
This point of intersection, (3 ; 4), is also illustrated graphically alongside.

Activity 3

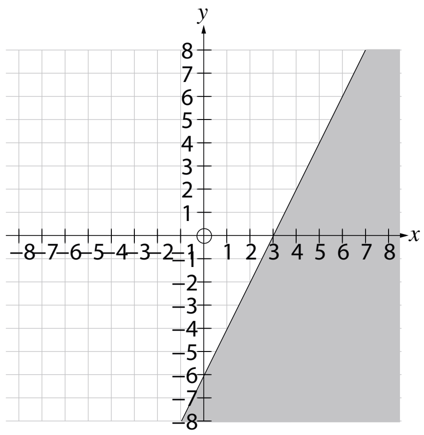
1. $x + y < 4$



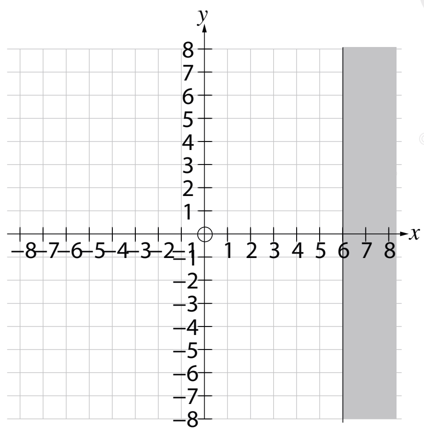
2. $4y - 2x > 10$



3. $4x - 2y \geq 12$

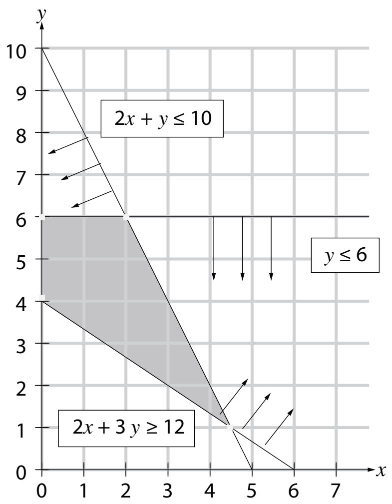


4. $x \geq 6$

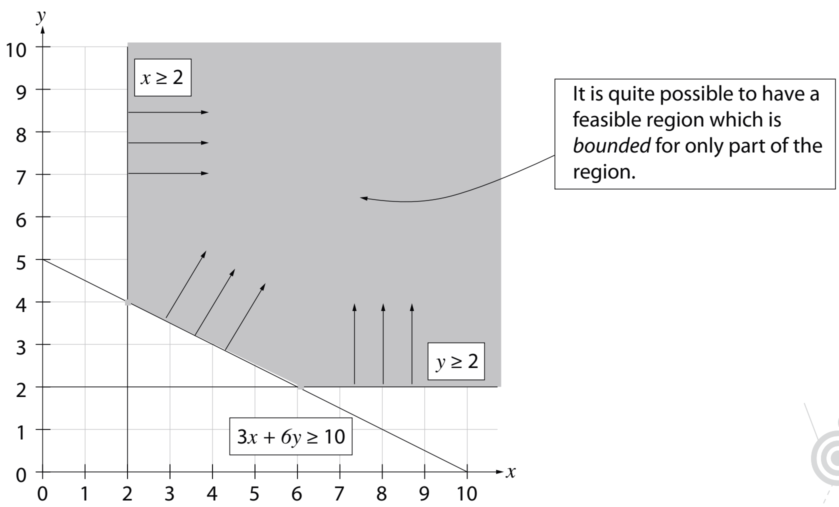


Activity 4

1. $y \leq 6$; $2x + y \leq 10$; $2x + 3y \geq 12$



2. $x \geq 2$; $y \geq 2$; $3x + 6y \geq 30$



Activity 5

1.1 $1\frac{1}{2}x + 2y \leq 24$ or $\frac{3}{2}x + 2y \leq 24$ (days)

1.2 $x + y \leq 14$

1.3 $4 \leq y$

Exclusive → $\frac{y}{x} \geq \frac{1}{2}$
Basic →
at least

> Always try to work with ratios here. "The number of Exclusive must be at least $\frac{1}{2}$ the number of Basic"

1.4 $\therefore 2y \geq x$

2. Number of hours available = 6 x 16 = 96 hours per week.

Total: 7hrs per single-seater

11hrs per 2-seater couch.

2.1 $\therefore 7x + 11y \leq 96$

2.2

Single-seater → $\frac{x}{y} \geq 2$
2-seater →
at least

$\therefore x \geq 2y$ or $2y \leq x$

2.3 $10 \leq x$

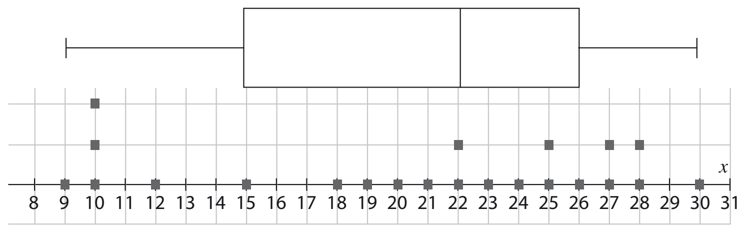
2.4 $y \leq 6$

Activity 6

1. $x + y \leq 40$; $10 \leq x \leq 30$; $10 \angle y$;

$3\ 000x + 6\ 000y \leq 180\ 000$ simplifies to $x + 2y \leq 60$

2.



3. Profit (P) = $600x + 1\ 000y$

4. Maximising the profit:

Using a table of values

Feasible Points	Objective Function P = $600x + 1\ 000y$	Optimal Solution (PROFIT)
A(10 ; 25)	P = 600(10) + 1 000(25)	R31 000
B(10 ; 10)	P = 600(10) + 1 000(10)	R16 000
C(30 ; 10)	P = 600(30) + 1 000(10)	R28 000
D(20 ; 20)	P = 600(20) + 1 000(20)	**R 32 000**

$\therefore$ the maximum profit occurs when the retailer sells 20 of type A and 20 of type B computers, yielding a profit of R32 000.