ANALYTICAL GEOMETRY

Revision of Grade 10 Analytical Geometry

Let's quickly have a look at the analytical geometry you learnt in Grade 10.

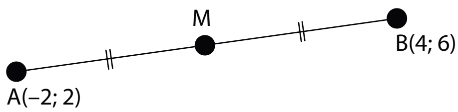
Midpoint formula

$$\left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$$

The midpoint formula is used to find the coordinates of the midpoint of a line segment.

Example

Find the midpoint of the line joining A(–2; 2) and B(4; 6)



Solution

To find the x co-ordinate of the midpoint we add the x coordinates of A and B together and divide by 2

$$\frac{x_1 + x_2}{2} = \frac{-2 + 4}{2}$$
$$= 1$$

We then do the same using the y values of A and B

$$\frac{y_1 + y_2}{2} = \frac{2 + 6}{2}$$
$$= 4$$

The coordinates of the mid-point are therefore M(1; 4).

Distance formula

The distance formula is used to find the distance between two points A(x_1 ; y_1) and B(x_2 ; y_2).

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

Find the distance between A(2; 3) and B(7; 4)

Solution

When using the distance formula we must remember that if we start with point A we must keep on using A first.

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$
$$= \sqrt{(2 - 7)^2 + (3 - 4)^2}$$

↑ ↑ ↑ ↑

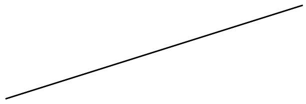
A B A B

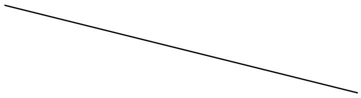
$$= \sqrt{(-5)^2 + (-1)^2}$$
$$= \sqrt{26}$$

The question might ask you to leave your answer in simplest surd form or to 1 or 2 decimal places.

Gradient (slope)

The gradient of a line gives us an indication of the steepness of the line. It is important to remember that a line with a positive gradient runs 'uphill' from left to right.





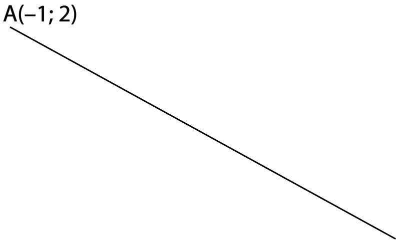
A line with a negative gradient runs 'downhill' if we look at it from left to right.

When calculating the gradient of a line we must always look at the answer and make sure it 'looks' right.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example

Calculate the gradient of the line AB.



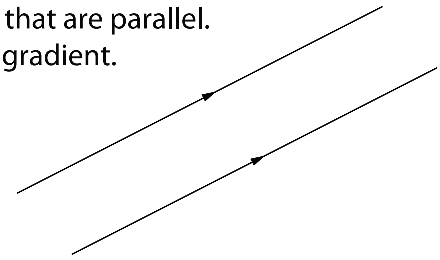
Solution

$$m_{AB} = \frac{2 - (-4)}{-1 - 3}$$
$$= \frac{6}{-4}$$
$$= -\frac{3}{2}$$

Our answer is negative and the line runs downhill from left to right so this looks correct.

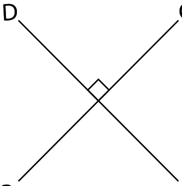
- Let's quickly talk about lines that are parallel.
  Parallel lines have the **same** gradient.

  $$m_1 = m_2$$



- What about perpendicular lines?
  Perpendicular lines intersect at an angle of 90°.
  For perpendicular lines

  $$m_1 \times m_2 = -1$$



This means if we multiply the 2 gradients with each other our answer must be –1.

eg. $m_{BC} = \frac{2}{3}$

$m_{DF} = -\frac{3}{2}$

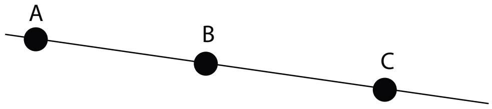
$$m_{BC} \times m_{DF} = \frac{2}{3} \times -\frac{3}{2}$$
$$= -1$$

The answer is –1 so the lines BC and DF must be perpendicular.

Colinear points

Do you still remember what colinear points are?

Colinear points are points that lie on the same straight line.



The points A, B and C lie on the same straight line and are therefore colinear. This means that $m_{AB} = m_{BC} = m_{AC}$. To prove that 3 points are colinear we only have to prove that 2 of the gradients are equal.

Now that we have revised what you learnt in Grade 10 we will use these skills to learn the following:

- **Finding the equation of a line.**
- **More about parallel and perpendicular lines.**
- **Inclination of a line.**

After we have gone through these topics we will then do an exercise where we test all of your analytical geometry skills!

Finding the equation of a line.

The equation of a straight line is given by $y = mx + c$

m is the gradient of the line and c is the y intercept

Example 1

Give the equation of a straight line with gradient = 4 and y-intercept –3.

Solution

This is easy, all we have to do is substitute m and c

$y = 4x - 3$

Example 2

Find the equation of the line with gradient = –4 passing through A(2; 8)

Solution

We use the formula $y - y_1 = m(x - x_1)$

$y - 8 = -4(x - 2)$

(x value of A) (y value of A) (substitute the gradient)

$y - 8 = -4x + 8$

$y = -4x + 16$

Example 3

Find the equation of the straight line joining A(–2; 3) and $B\left(1; \frac{3}{2}\right)$

Solution

Solution

First we must calculate the gradient between A and B using the gradient formula

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{3 - \frac{3}{2}}{-2 - 1}$$

$$= \frac{\frac{3}{2}}{-3}$$

$$= -\frac{1}{2}$$

then we use the formula $y - y_1 = m(x - x_1)$

Into this formula we substitute the gradient and one of our 2 points.

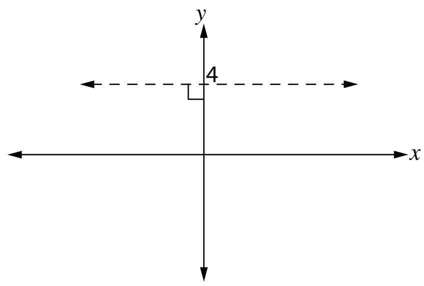
I will use A(−2; 3)

$$y - 3 = -\frac{1}{2}(x - (-2))$$

$$y - 3 = -\frac{1}{2}x - 1$$

$$y = -\frac{1}{2} + 2$$

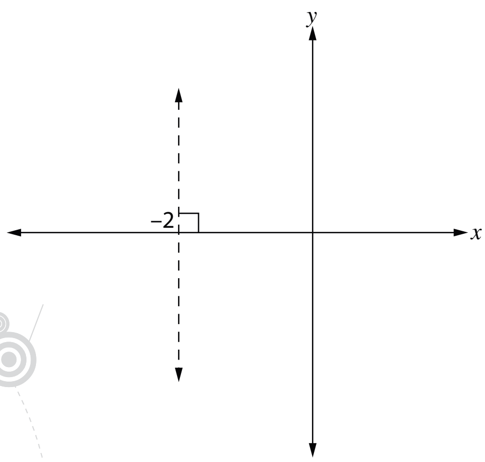
Special lines



Consider the line above:

This line is parallel to the x axis and intersects the y axis at 4.

These lines have the general equation $y = c$ where c is the y intercept. The equation of the line is therefore $y = 4$



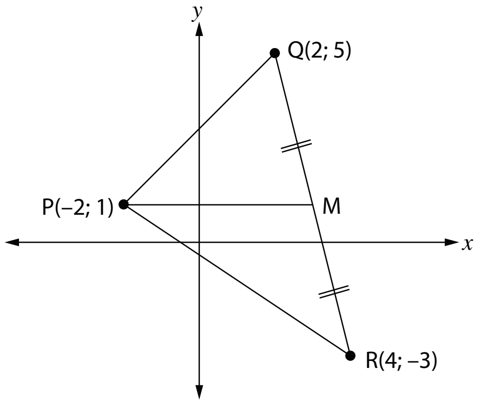
The line above is parallel to the y axis and intersects the x axis at −2. These lines have the general equation $x = c$ where c is the x intercept. The equation of the line is therefore $x = -2$.

Example 4

P(–2; 1) Q(2; 5) and R(4; –3) are the vertices of △PQR. M is the midpoint of QR. Determine the equation of line PM.

Solution

It is always helpful to draw a rough diagram.



We first have to find the coordinates of M. M is the midpoint of the line QR so we use the midpoint formula

$$\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{2 + 4}{2}; \frac{5 + (-3)}{2}\right)$$

M(3; 1)

Now that we have the coordinates of M we use the gradient formula to calculate the gradient of PM.

$$m_{PM} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1 - 1}{-2 - 3}$$

$$= \frac{0}{-5} = 0$$

This is interesting!!

What does it mean if our gradient is zero?

It means the line is parallel to the x-axis. Therefore the equation is $y = 1$. (Please note that had the gradient not been zero we would have used “the formula” $y - y_1 = m(x - x_1)$ to calculate the equation in the form $y = mx + c$.

Parallel and perpendicular lines

We revised earlier that parallel lines have the same gradient and perpendicular lines have gradients with product equal to – 1.

Example 5

Show that the line AB defined by $y = 2x + 3$ is perpendicular to the line CD, defined by $2y + x - 2 = 0$

Solution

- Both equations must be in standard form.

$y = 2x + 3$ $\qquad$ $2y + x - 2 = 0$

$$2y = -x + 2$$

$$y = -\tfrac{1}{2}x + 1$$

Then compare the gradients

$$m_{AB} = 2 \qquad m_{CD} = -\frac{1}{2}$$

$$m_{AB} \times m_{CD} = -1$$

Therefore the lines AB and CD are perpendicular.

Example

Example 6

Determine the equation of the line AB if it is given that AB is perpendicular to the line $2y = 4x + 8$ and passes through (–2; –8).

Solution

Solution

Write the equation of the given line in standard form.

$$2y = 4x + 8$$

$$y = 2x + 4$$

The line has a gradient of +2. This means that the gradient of AB $= -\frac{1}{2}$ as it is given that the lines are perpendicular.

Now we use the formula $y - y_1 = m(x - x_1)$ to determine the equation of the line.

$$y - (-8) = -\frac{1}{2}(x - (-2))$$

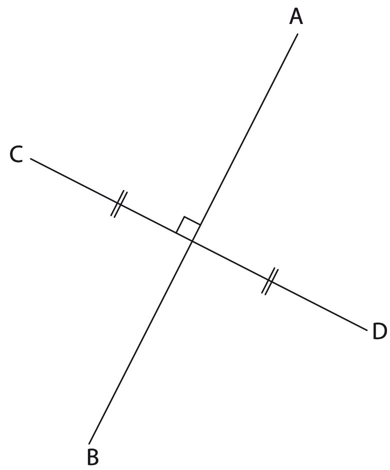
$$y + 8 = -\frac{1}{2}x - 1$$

$$y = -\frac{1}{2}x - 9$$

Perpendicular bisectors

It is important to know the properties of a perpendicular bisector. The name tells us all we need to know!

In the diagram alongside the line AB is the perpendicular bisector of CD. AB is perpendicular to CD and it cuts CD into 2 equal parts. (Passes through the midpoint).



Example

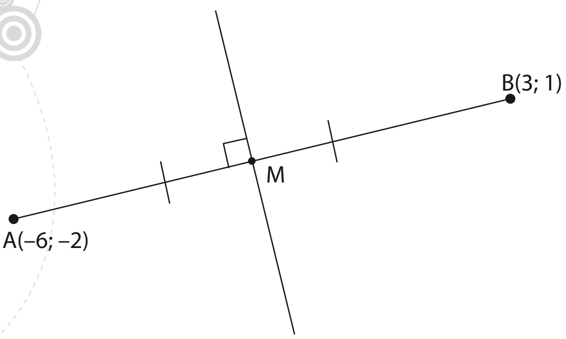
Example 7

Find the equation of the perpendicular bisector of AB if A(–6; –2) and B(3; 1).

Solution

Solution

As always it is a good idea to make a rough sketch.



First we calculate the coordinates of M the midpoint of AB as our perpendicular bisector passes throught the point M. We use the midpoint formula.

$$\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{-6 + 3}{2}; \frac{-2 + 1}{2}\right)$$

$$M\left(\frac{-3}{2}; \frac{-1}{2}\right)$$

Now we calculate the gradient of the line AB.

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 1}{-6 - 3} \\ &= \frac{-3}{-9} \\ &= \frac{1}{3} \end{aligned}$$

We know that our perpendicular bisector must have a gradient of –3 as $m_1 \times m_2 = -1$ for perpendicular lines.

We now use the coordinates of the midpoint as well as the gradient to find the equation of the perpendicular bisector.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \left(-\tfrac{1}{2}\right) &= -3\left(x + \tfrac{3}{2}\right) \\ y + \tfrac{1}{2} &= -3x - \tfrac{9}{2} \\ y &= -3x - 5 \end{aligned}$$

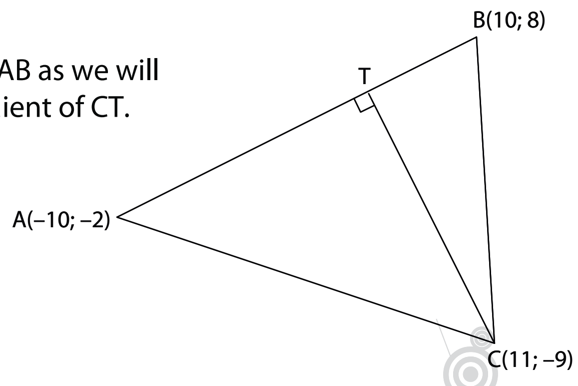
$$\begin{aligned} y - \left(-\tfrac{1}{2}\right) &= -3\left[x - \left(-\tfrac{3}{2}\right)\right] \\ y + \tfrac{1}{2} &= -3\left[x + \tfrac{3}{2}\right] \\ y + \tfrac{1}{2} &= -3x - \tfrac{9}{2} \qquad (\times 2) \\ 2y + 1 &= 6x - 9 \\ 2y + 1 &= -6x - 9 \\ 2y &= -6x - 10 \\ \therefore\ y &= -3x - 5 \end{aligned}$$

Example 8

A(–10; –2) B(10; 8) and C(11; –9) are the vertices of $\triangle$ABC. Determine the equation of CT If CT is perpendicular to AB.

Solution

First we determine the gradient of AB as we will then be able to determine the gradient of CT.



$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 8}{-10 - 10} \\ &= \frac{-10}{-20} \\ &= \frac{1}{2} \end{aligned}$$

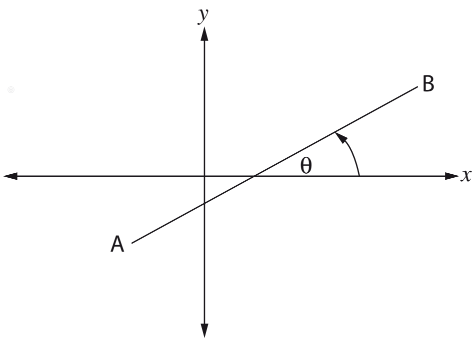
$\therefore$ the gradient of CT = –2.

We now use C(11; –9) and $m = -2$ to determine the equation of CT.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-9) &= -2(x - 11) \\ y + 9 &= -2x + 22 \\ y &= -2x + 13 \end{aligned}$$

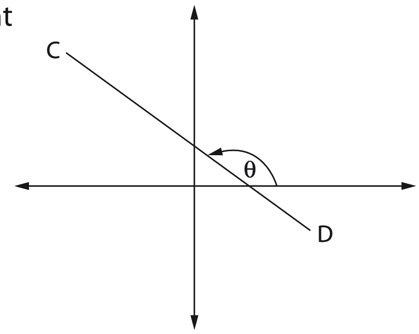
Angle of Inclination

The angle of inclination of a line is the angle that the line makes with the positive x-axis.



Consider the line AB above. AB has a positive gradient as the line runs 'uphill' from left to right. The angle of inclination θ is an acute angle.

Consider the line CD. CD has a negative gradient as CD runs 'downhill' from left to right. The angle of inclination θ is an obtuse angle.



The size of the angle of inclination can be calculated using

$\tan\theta = m$

$m = \frac{y}{x}$

$\tan\theta = \frac{y}{x}$

$\therefore \tan\theta = m$

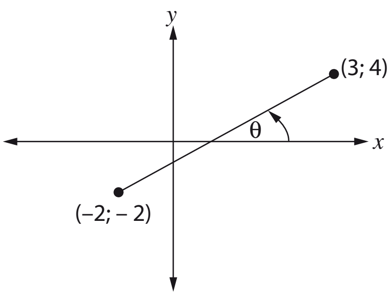
Example

Example 9

Calculate the angle which the line passing through (–2; –2) and (3; 4) makes with the positive x axis.

Solution

Solution



First we calculate the gradient of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4-(-2)}{3-(-2)} = \frac{6}{5}$$

The line has a positive gradient so we know that θ will be an acute angle.

$\tan\theta = m$

$\tan\theta = \frac{6}{5}$

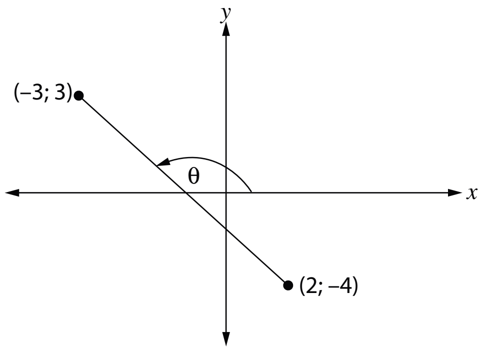
We calculate θ by using the $\tan^{-1}$ function on our calculator.

$\theta = 50{,}2°$

Example 10

Calculate the angle between the line passing through (–3; 3) and (2; –4) and the positive x axis.

Solution



Again we start by calculating the gradient of our line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3 - (-4)}{-3 - 2}$$
$$= \frac{7}{-5}$$

The line has a negative gradient so we know that θ will be an obtuse angle.

$$\tan\theta = m$$

$$\tan\theta = \frac{-7}{5}$$

We calculate a reference angle by using the $\tan^{-1}$ function on our calculator.

Ref angle = –54,5°

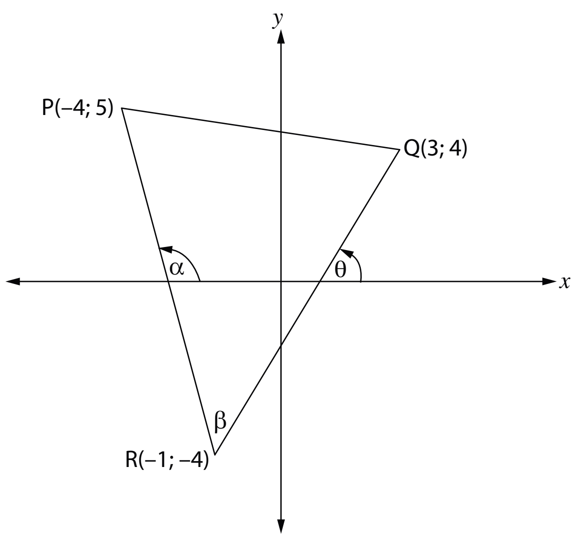
Now we add 180° to give us an angle between 90° and 180°.

$$\theta = -54{,}5° + 180°$$
$$= 125{,}5°$$

We only add 180° if our gradient is negative!!!!!

Let’s have a look at a slightly more difficult example.

Example 11



Calculate the size of angle β.

Solution

Solution

In order to calculate the size of angle B we will first have to calculate α and θ.

$$m_{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-4)}{-4-(-1)} = \frac{9}{-3} = -3$$

$$m_{QR} = \frac{4-(-4)}{3-(-1)} = 2$$

$\tan\theta = 2$

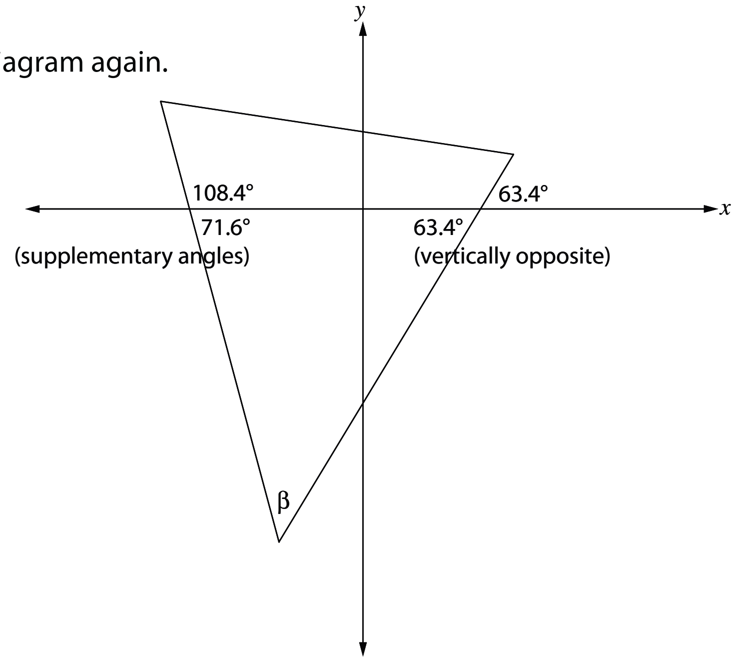
$\theta = 63{,}4°$

$\tan\alpha = -3$

Ref $\angle = -71{,}6°$

$\alpha = 108{,}4°$

Now let's look at the diagram again.



$\angle B = 180° - (71{,}6° + 63{,}4°)$

$= 45°$ (angles of a triangle add up to 180°)

Activity

Activity

Practise the questions in this section to ensure you understand all the concepts in Analytical geometry.

1. Determine the equation of the straight line that passes through:

1.1 the points A(3; 4) and B(1; 5)

1.2 the point (–6; –2) and perpendicular to $3x + 2y = 6$

1.3 (3; –1) and parallel to $2y - 6x = 1$

2. A(–5; 5), B(4; –5) and C(1; p) are colinear. Determine

2.1 the length of AB

2.2 the gradient of AB

2.3 the angle AB makes with the positive x axis

2.4 the value of p.

3. Given A(–5; 1), B(1; 6) and C(7; – 2). Determine

3.1 the length of AC

3.2 the equation of the line BC

3.3 the coordinates of P the midpoint of AB

3.4 the equation of the line parallel to AC passing through the point (–1; 3)

4. Determine the equation of the line through the point (–2; –3) and perpendicular to the line $2x + 3y = 13$.

5. A(–4; 3), B(6; –2), T(8; a) and S(4; t) and points in a cartesian plane.

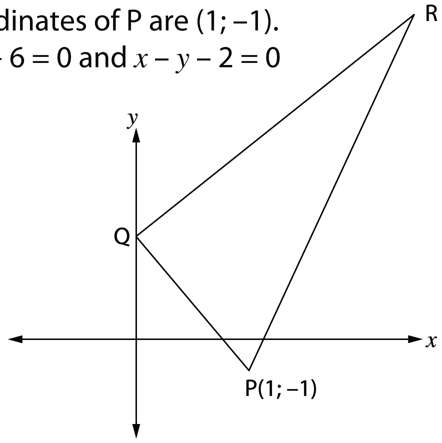
5.1 Determine the equation of the line AB.

5.2 Calculate the value of a if BT⊥AB

5.3 Calculate the value of b if M(b; 0) is the midpoint of BT

5.4 Calculate the values of t if BS = $2\sqrt{10}$

6. In the figure PQR is a triangle. The coordinates of P are (1; –1). The equations of QR and PR are $x - 3y + 6 = 0$ and $x - y - 2 = 0$ respectively.



6.1 Show that coordinates of Q are (0; 2)

6.2 Calculate the gradient of QR

6.3 Prove that $P\hat{Q}R = 90°$

6.4 Find the coordinates of R

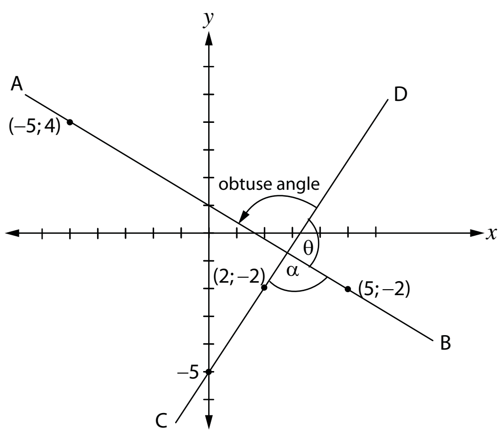
6.5 Calculate the length of RQ

7. Calculate the gradient of the line with inclination 100°

8. Find the equation of the perpendicular bisector of MN if M is the point (–3; 2) and N is the point (5; –8)

9. ABCD is a parallelogram with vertices A(2; 3), B(7; –5), C(3; –11) and D(x; y). Find the 4[th] vertex of the paralellogram.

10. Find the obtuse angle between AB and CD IF A(–5; 4), B(5; –2) and the equation of CD is $2y - 3x + 10 = 0$



Solutions

1.1 $m_{AB} = \frac{4-5}{3-1} = \frac{-1}{2}$

$y - y_1 = m(x - x_1)$

$(y - 4) = -\frac{1}{2}(x - 3)$

$2y - 8 = -x + 3$

$2y = -x + 11$

$y = -\frac{1}{2}x + \frac{11}{2}$

1.2 $3x + 2y = 6$

$2y = -3x + 6$

$y = -\frac{3}{2}x + 3$

$m = -\frac{3}{2}$

m of $\perp$ line $= \frac{2}{3}$

$y - y_1 = m(x - x_1)$

$y - (-2) = \frac{2}{3}(x - (-6))$

(×3) $y + 2 = \frac{2}{3}(x + 6)$

$3y + 6 = 2(x + 6)$

$3y + 6 = 2x + 12$

$3y = 2x + 6$

$y = \frac{2}{3}x + 2$

1.3 $2y - 6x = 1$

$2y = 6x + 1$

$y = 3x + \frac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - (-1) = 3(x - 3)$

$y + 1 = 3x - 9$

$y = 3x - 10$

2..1 $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(-5-4)^2 + (5-(-5))^2}$

$= \sqrt{81 + 100}$

$= \sqrt{181}$

2.2 $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{5-(-5)}{-5-4}$

$= \frac{10}{-9}$

2.3 $\tan\theta = m$

$\tan\theta = \frac{-10}{9}$

Ref $\theta = -48$

$\theta = 132°$

2.4 $m_{BC} = m_{AB}$

$$\frac{p+5}{1-4} = \frac{-10}{9}$$

$$9p + 45 = 30$$

$$p = \frac{-15}{9}$$

$$= \frac{-5}{3}$$

3.1 $AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(-5-7)^2 + (1-(-2))^2}$$

$$= \sqrt{144 + 9}$$

$$= \sqrt{153}$$

3.2 First calculate the gradient of BC

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-(-2)}{1-7}$$

$$= \frac{8}{-6}$$

$$= \frac{-4}{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - 6 = -\frac{4}{3}(x-1)$$

$$3y - 18 = 4x + 4$$

$$3y = -4x + 22$$

$$y = -\frac{4}{3}x + \frac{22}{3}$$

3.3 $\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

$$\left(\frac{-5+1}{2}; \frac{1+6}{2}\right)$$

$$P\left(-2; \frac{7}{2}\right)$$

3.4 $m_{AC} = \frac{-2-1}{7-(-5)} = \frac{-3}{12}$

$$= \frac{-1}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{-1}{4}(x - (-1))$$

$$4y - 12 = -x - 1$$

$$4y = -x + 11$$

$$y = \frac{-1}{4}x + \frac{11}{4}$$

4. First write equation of line in standard form

$$3y = -2x + 13$$

$$y = \frac{-2}{3}x + \frac{13}{3}$$

gradient $\therefore \perp$ line gradient $m = \frac{3}{2}$

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = \frac{3}{2}(x - (-2))$$

$$y + 3 = \frac{3}{2}(x + 2)$$

$$2y + 6 = 3x + 6$$

$2y = 3x$

$y = \frac{3}{2}x$

5.1 $m_{AB} = \frac{3-(-2)}{-4-6} = \frac{5}{-10} = -\frac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - 3 = -\frac{1}{2}(x - (-4))$

$2y - 6 = -x - 4$

$2y = -x + 2$

$y = -\frac{1}{2}x + 1$

5.2 $m_{AB} = \frac{-1}{2}$

$\therefore m_{BT} = 2$ if AB⊥BT

$m_{BT} = \frac{a-(-2)}{8-6} = 2$

$\frac{a+2}{2} = \frac{2}{1}$

$a + 2 = 4$

$a = 2$

5.3 $\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

$\frac{6+8}{2} = b$

$2b = 14$

$b = 7$

5.4 $BS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$BS = \sqrt{(6-4)^2 + (-2-t)^2}$

$= \sqrt{4 + 4 + 4t + t^2}$

$= \sqrt{t^2 + 4t + 8}$

$BS^2 = t^2 + 4t + 8$

$\therefore (2\sqrt{10})^2 = t^2 + 4t + 8$

$0 = t^2 + 4t - 32$

$(t + 8)(t - 4) = 0$

$\therefore t = -8$ or $t = 4$

6.1 First put both equations into std form.

$x - y - 2 = 0$

$-y = -x + 2$

$y = x - 2$ Must be RP ($-y$ intercept)

$-3y = -x - 6$

$y = \frac{1}{3}x + 2$ Must be QR ($+y$ intercept)

y-intercept = 2 ($x = 0$)

$\therefore$ Q = (0; 2)

6.2 $m_{QR} = \frac{1}{3}$ (see 6.1)

6.3 $m_{QP} = \frac{y_2 - y_1}{x_2 - x_1}$ Use Q

$= \frac{2 - (-1)}{0 - 1}$

$= -3$

$m_{QR} \times m_{QP} = \frac{1}{3}(-3) = -1$

$\therefore\ P\hat{Q}R = 90°$

6.4 R is the point of intersection of QR and RP

$\frac{1}{3}x + 2 = x - 2$

$x + 6 = 3x - 6$

$-2x = -12$

$x = 6$

to find y value sub $x = 6$ back into either equation

$y = (6) - 2$

$= 4$

R(6; 4)

6.5 $RQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(6 - 0)^2 + (4 - 2)^2}$

$= \sqrt{40}$

7. $\theta = 100°$

$\therefore \tan 100° = m$

$\therefore m = -5{,}67$

8.1 $y = 3x + 2$

$\therefore m = 3$

$y - y_1 = m(x - x_1)$

$y + 2 = 3(x - 4)$

$y = 3x - 14$

8.2 $m_{PQ} = \frac{1 - (-3)}{7 - (-1)}$

$= \frac{4}{8}$

$= \frac{1}{2}$

$\therefore m\perp = -2$

P(–1; –3) sub into $y - y_1 = m\,(x - x_1)$

$\therefore\ y + 3 = -2(x + 1)$

$y = -2x - 2 - 3$

$y = -2x - 5$

8.3 $m_{MN} = \frac{-8 - 2}{5 - (-3)}$

$= \frac{-10}{8}$

$= -\frac{5}{4}$

$m\perp = \frac{4}{5}$

Midpoint MN $\left(\frac{-3 + 5}{2}; \frac{2 - 8}{2}\right)$

$(1; -3)$

$\therefore\ y - y_1 = m(n - n_1) \Rightarrow y - (-3) = \frac{4}{5}(x - 1)$

$5y + 15 = 4n - 4$

$5y = 4n - 19$

$y = \frac{4}{5}n - \frac{19}{5}$

9. Midpoint AC $\left(\frac{2+3}{2}; \frac{3-11}{2}\right)$

$\left(\frac{5}{2}; -4\right)$

Midpoint BD $\left(\frac{7+x}{2}; \frac{-5+y}{2}\right)$

$\therefore \frac{7+x}{2} = \frac{5}{2}$

$\therefore x = -2$

$\frac{-5+y}{2} = -4$

$-5 + y = -8$

$y = -3$

10. $m_{AB} = \frac{-2-4}{5+5}$

$= \frac{-6}{10}$

$= \frac{-3}{5}$

$\tan\theta = \frac{-3}{5}$

$\therefore \theta = 149{,}04°$

CD: $2y = 3x - 10$

$\therefore y = \frac{3}{2}x - 5$

$\therefore m_{CD} = \frac{3}{2}$

$\tan\alpha = \frac{3}{2}$

$\therefore \alpha = 56{,}31°$

$\therefore \angle$ between AB and CD $= 149{,}04 - 56{,}31$

$= 92{,}73°$