TRANSFORMATION GEOMETRY

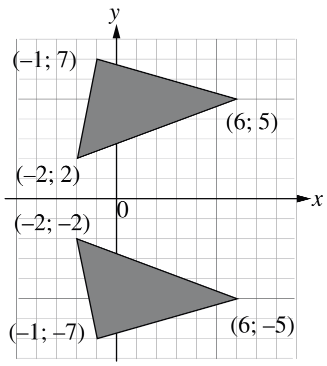
In this section we will be looking at the following Geometric transformations:

- Rotating points about the origin clockwise through 90°
- Rotating points about the origin anti-clockwise through 90°
- Rotating through 180°
- Enlargements through the origin

Grade 10 Reflections

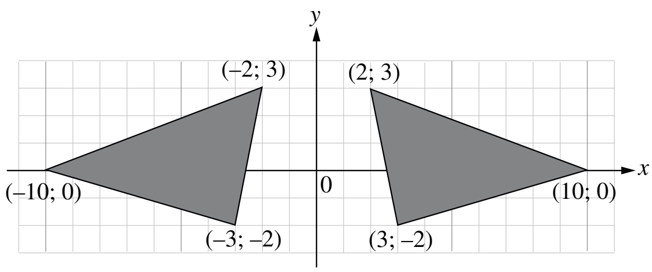
In Grade 10, you studied reflections (flips) about the x-axis and the y-axis .

The diagrams below illustrate these reflections.



Reflection about x-axis:

Rule: $(x; y) \rightarrow (x; -y)$

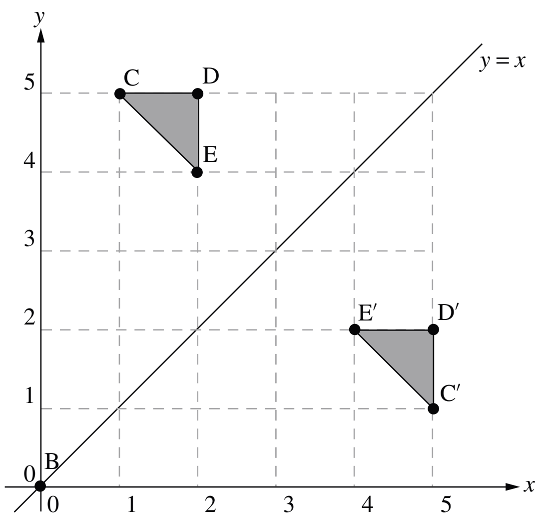


Reflection about y-axis:

Rule: $(x; y) \rightarrow (-x; y)$

Also in Grade 10, you studied reflections about the line $y = x$.

The diagram below illustrates this reflection.



Reflection about the line $y = x$

Rule: $(x; y) \rightarrow (y; x)$

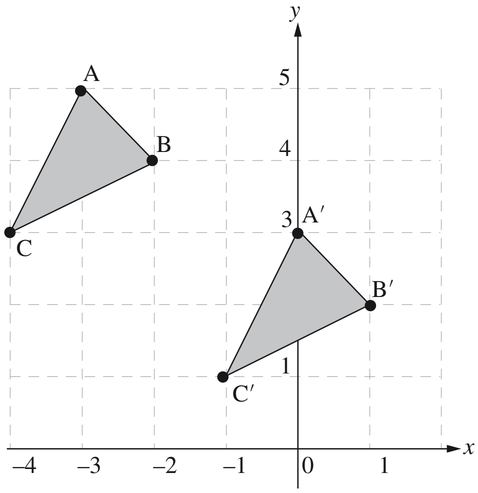
Notice how C(1; 5) $\rightarrow$ C′(5; 1), D(2; 5) $\rightarrow$ D′(5; 2) and E(2; 4) $\rightarrow$ E′(4; 2)

Grade 10 Translations

In Grade 10, you studied the translations (slides) of shapes in the Cartesian Plane.

Suppose that $\triangle ABC$ is translated 3 units horizontally to the right and 2 units vertically down.

$\triangle ABC$, together with its image denoted by $\triangle A'B'C'$ are drawn below.



Notice that 3 has been added to the x value at each of A, B and C to get the new x value at A′, B′ and C′.

$x_A = -3$ and $x_{A'} = 0$; $x_B = -2$ and $x_{B'} = 1$; $x_C = -4$ and $x_{C'} = -1$

Also 2 has been subtracted from the y at each of A, B and C to get the new y value at A′, B′ and C′.

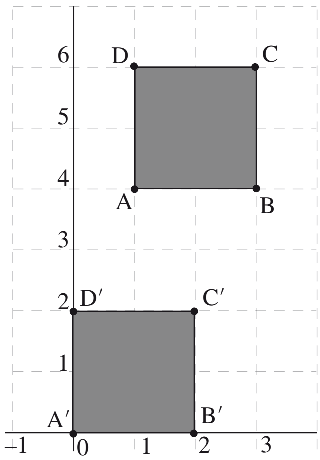
$y_A = 5$ and $y_{A'} = 3$; $y_B = 4$ and $y_{B'} = 2$; $y_C = 3$ and $y_{C'} = 1$

In general, the rule applied is $(x; y) \rightarrow (x + 3; y - 2)$

Example

Example

Study the diagram below illustrating a translation of rectangle ABCD. Can you write down the general rule that applies?



Answer: $(x; y) \rightarrow (x - 1; y - 4)$

Grade 11 Transformations

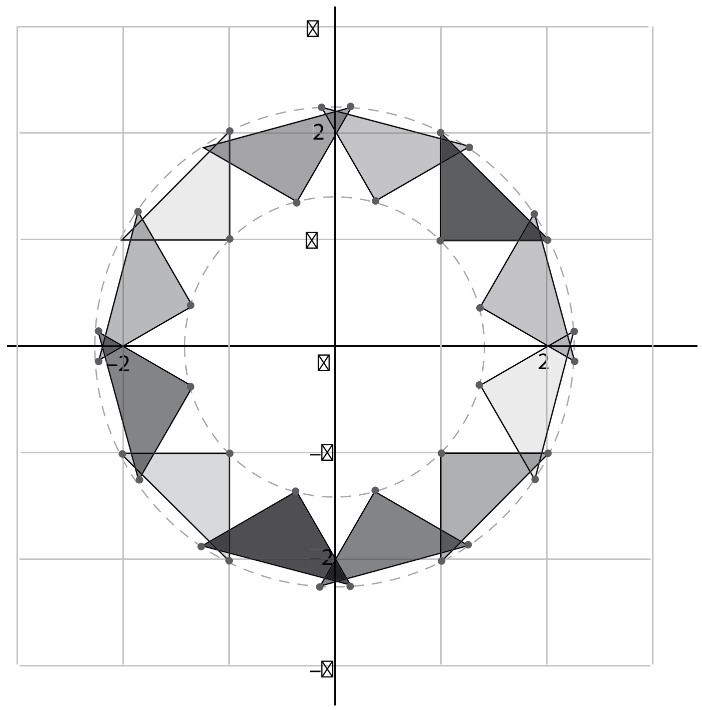
In grade 11, you study the following two types of transformations: rotations and enlargements.

Rotations

To rotate is to turn the shape around a point. Our curriculum requires that we rotate about the origin (0; 0).

When we rotate an object about a point, the distance from the point to the object stays the same.

This is illustrated below:



The diagram above shows that as we rotate the triangle about the origin, the distance from the origin to the points on the triangle stays constant.

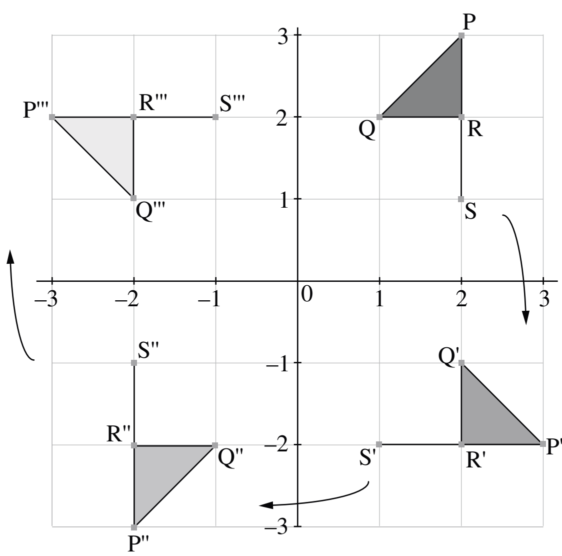
In grade 11, we look at rotations of 90° in clockwise and anticlockwise directions about the origin, as well as rotations of 180° about the origin.

Rotating in a clockwise direction through 90° about the origin

Example

Example

Consider the flag, PQRS, given in the diagram below. Each time, the flag is rotated about the origin through an angle of 90° in a clockwise direction.



Notice how the coordinates of P change as we rotate it about the origin.

$P(2; 3) \rightarrow P'(3; -2)$

$P'(3; -2) \rightarrow P''(-2; -3)$

$P''(-2; -3) \rightarrow P'''(-3; 2)$

In general, every time the shape is rotated through **90°** in a **clockwise** direction, the coordinates change according to the rule: $(x; y) \rightarrow (y; -x)$

Write down, using the general rule, the coordinates of S, S′, S″ and S‴.

Check your answers by reading off the coordinates from the diagram.

Answer:

$S(2; 1) \rightarrow S'(1; -2)$

$S'(1; -2) \rightarrow S''(-2; -1)$

$S''(-2; -1) \rightarrow S'''(-1; 2)$

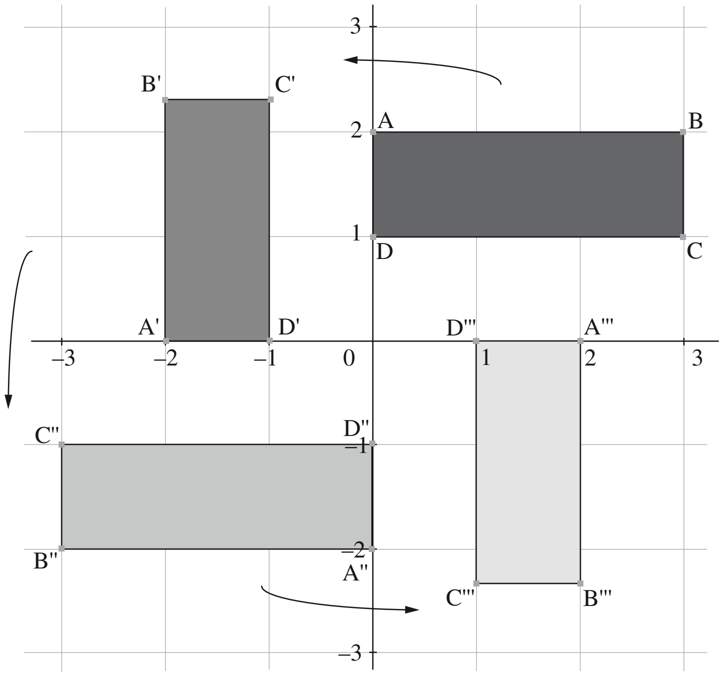
Do the same for the coordinates of Q and R.

Rotating in an anticlockwise direction through 90° about the origin

Example

Example

Consider the rectangle, ABCD, given in the diagram below. Each time, the rectangle is rotated about the origin through an angle of 90° in an anticlockwise direction.



Notice how the coordinates of B change as we rotate it about the origin.

$B(3; 2) \rightarrow B'(-2; 3)$

$B'(-2; 3) \rightarrow B''(-3; -2)$

$B''(-3; -2) \rightarrow B'''(2; -3)$

In general, every time the shape is rotated through **90°** in an **anticlockwise** direction, the coordinates change according to the rule: $(x; y) \rightarrow (-y; x)$

Write down, using the general rule, the coordinates of C, C′, C″ and C‴.

Check your answers by reading off the coordinates from the diagram.

Answer

C(3; 1) $\rightarrow$ C′(–1; 3)

C′(–1; 3) $\rightarrow$ C″(–3; –1)

C″(–3; –1) $\rightarrow$ C‴(1; –3)

Do the same for the coordinates of A and D.

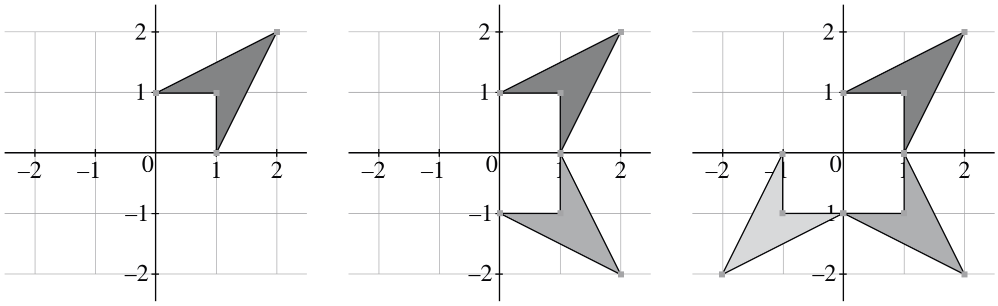
Rotating through 180° about the origin

Rotating an object about the origin through 180° results in the same image as the following combination of reflections:

First reflect in the y-axis and then reflect the image in the x-axis or vice versa.

Consider the shape PQRS below:

The first image shows the reflection of PQRS in the x-axis. The resulting image is then reflected about the y axis.



The final image is the same as the image obtained if the shape is rotated about the origin through 180°.

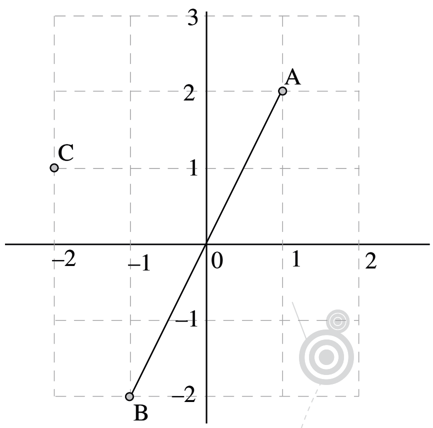
In general,

Shape reflected in x-axis: $(x; y) \rightarrow (x; -y)$
Image reflected in y-axis: $(x; -y) \rightarrow (-x; -y)$

The rotation of 180° about the origin can also be viewed as a double rotation of 90° in the same direction:

See the diagram below:
C is obtained by rotating A(1; 2) in an anticlockwise direction through 90°. Hence C(–2; 1). C is then rotated again through 90° in an anticlockwise direction to B(–1; –2).

A(1; 2) $\rightarrow$ C(–2; 1)
C(–2; 1) $\rightarrow$ B(–1; –2)



In general, $(x; y) \rightarrow (-y; x)$ is applied twice.

Summary of Rules for Rotations in grade 11

Rotate through 90° in clockwise direction: $(x; y) \rightarrow (y; -x)$

Rotate through 90° in anticlockwise direction: $(x; y) \rightarrow (-y; x)$

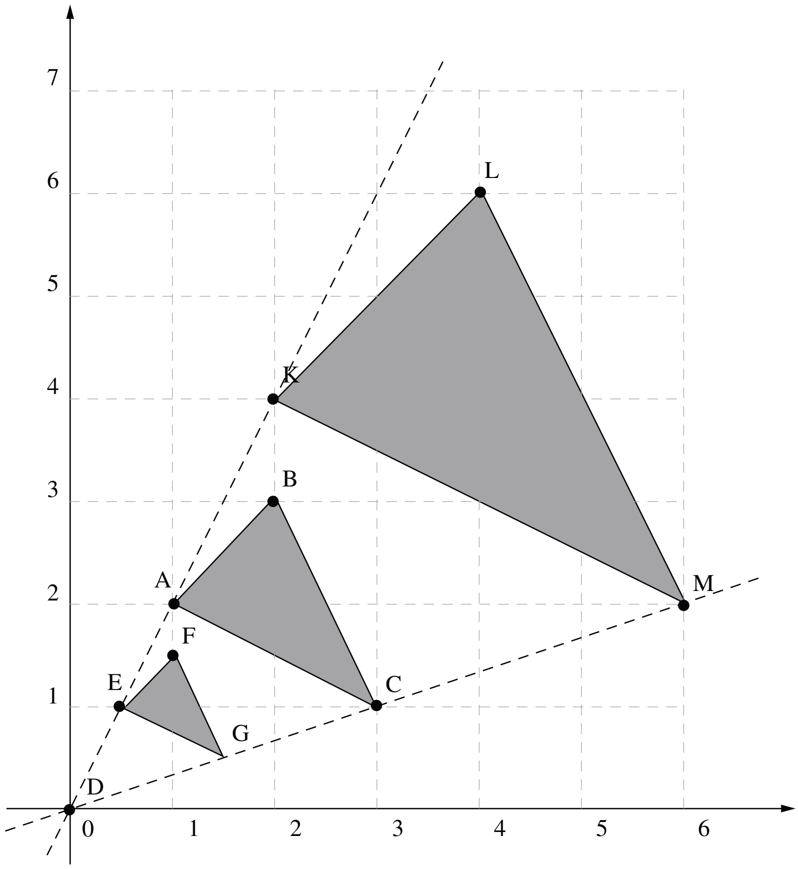
Rotate through 180°: $(x; y) \rightarrow (-x; -y)$

Enlargements through the origin

The second type of transformation that must be known in grade 11 is that of an enlargement through the origin.

With an *enlargement* the shape becomes bigger or smaller. Enlargements of factor greater than 1 will increase the size of the shape. Enlargements of factor smaller than 1 and positive will decrease the size of the original shape.

This is illustrated below:



$\triangle$EFG is an *enlargement* of $\triangle$ABC through the origin by a factor of $\frac{1}{2}$.

Notice that for each pair of coordinates, the rule $(x; y) \rightarrow \left(\frac{1}{2}x; \frac{1}{2}y\right)$ is applied.

For example, A(1; 2) $\rightarrow$ E$\left(\frac{1}{2}; 1\right)$; B(2; 3) $\rightarrow$ F$\left(1; \frac{3}{2}\right)$ and C(3; 1) $\rightarrow$ G$\left(\frac{3}{2}; \frac{1}{2}\right)$

$\triangle$KLM is an enlargement of $\triangle$ABC through the origin by a factor of 2.

Notice that for each pair of coordinates, the rule $(x; y) \rightarrow (2x; 2y)$ is applied.

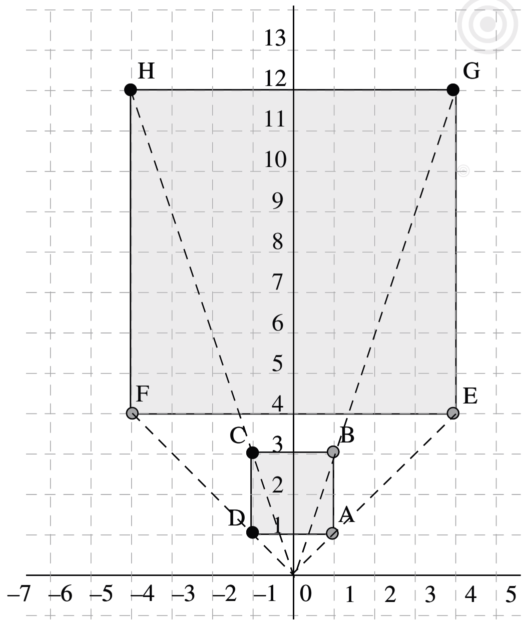
For example, A(1; 2) $\rightarrow$ K(2; 4) ; B(2; 3) $\rightarrow$ L(4; 6) and C(3; 1) $\rightarrow$ M(6; 2)

Enlargements produce similar shapes and therefore the interior angles of a shape will remain constant.

Important fact: If a shape is enlarged by a factor of k through the origin, then the Area of the new image changes by a factor of k^2.

Consider the following example:

1. Square ABCD has been enlarged through the origin by a factor of k. The resulting image is EGHF. What is the value of k?
2. Calculate the area of ABCD.
3. Calculate the area of EFGH.
4. By what factor has the area of EFGH increased compared to ABCD?



Answers:

1. Each of the points on ABCD have been transformed by the rule:

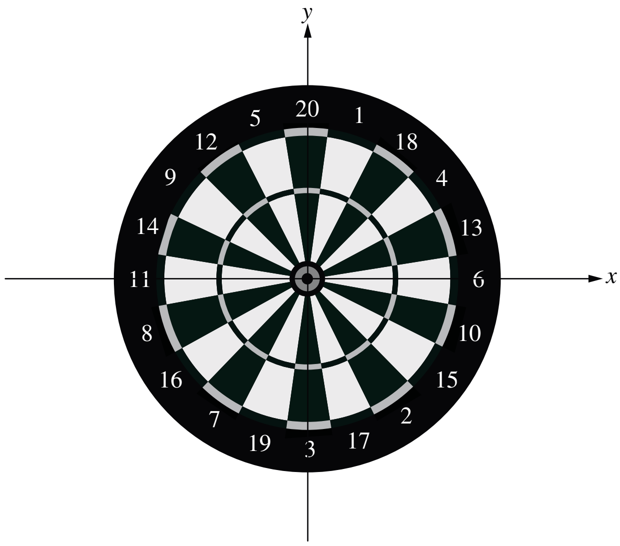
   $(x; y) \longrightarrow (4x; 4y)$. Therefore $k = 4$.
2. Area of ABCD $= 2^2 = 4$
3. Area of EGHF $= 8^2 = 64$
4. $\frac{\text{Area EGFH}}{\text{Area ABCD}} = \frac{64}{4} = 16.$

   Therefore the area has been increased by a factor of $16 = 4^2$

Activity

Activity

1. Suppose that the depicted dart board is placed in the Cartesian plane with the bulls eye (centre of the board) placed at the origin.



1.1 If the number 20 is rotated about the origin through 180°, write down the number that it will coincide with on the board.

....

1.2 If the number 11 is rotated clockwise about the origin through an angle of 90°, write down the number on the board that it would coincide with.

....

1.3 If the number 13 is rotated about the origin through 180°, write down the number it will coincide with on the board.

....

1.4 If the number 2 is positioned at point P(a; b), give, in terms of a and b, the coordinates of the point where the number 12 is positioned.

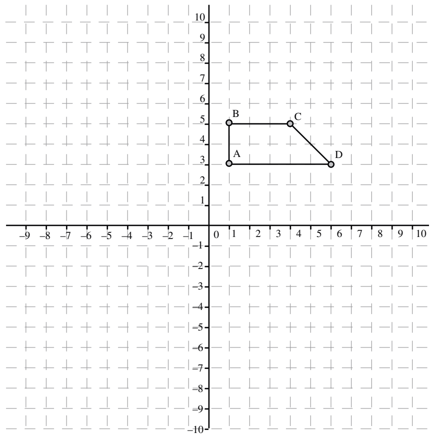
....

1.5 If the number 19 is positioned at the point Q(m; n), give, in terms of m and n, the coordinates of the point where the number 14 is positioned.

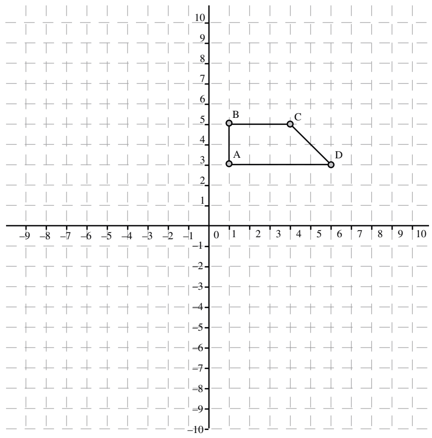
....

2. In each case, draw the image of ABCD which is obtained from the given transformation. Draw the image on the grid provided.

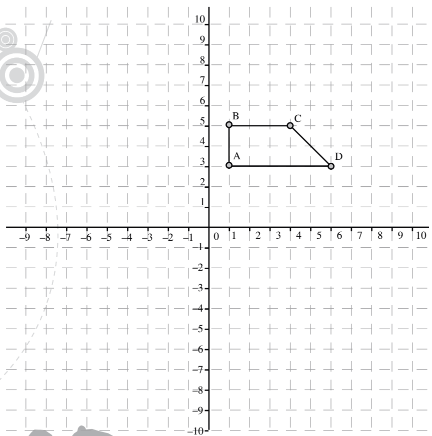
2.1 ABCD is enlarged by a factor of $\frac{1}{2}$.



2.2 ABCD is rotated clockwise about the origin through 90°



2.3 ABCD is rotated through an angle of 180° about the origin.

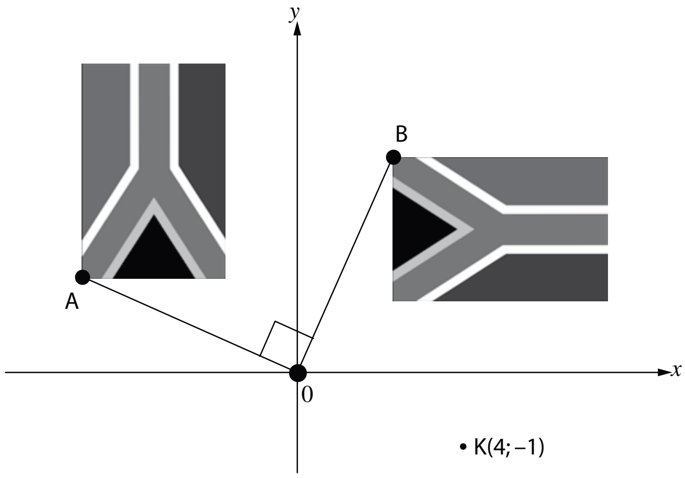


3. In each case the South African flag, positioned in the first quadrant, undergoes a transformation as illustrated in the diagram given. In each case vertex B goes to A.

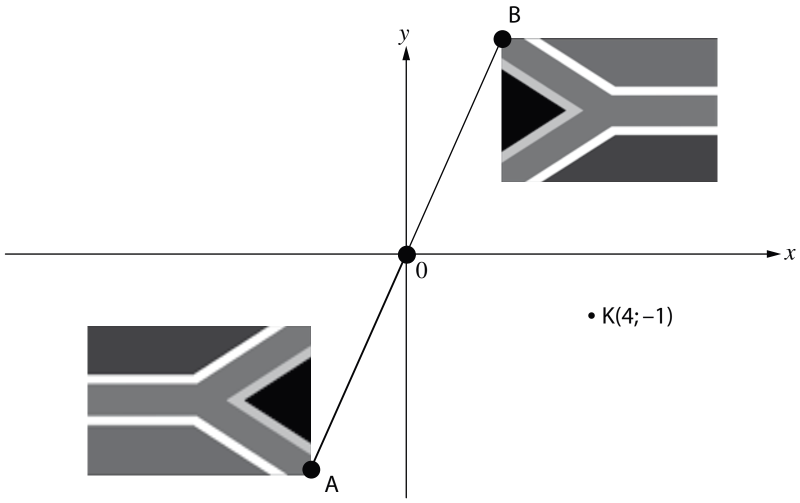
3.1 In each case give the coordinates of the image of K if it has to undergo the same transformation as the South African flag.

3.2 give a description and the general rule that was applied.

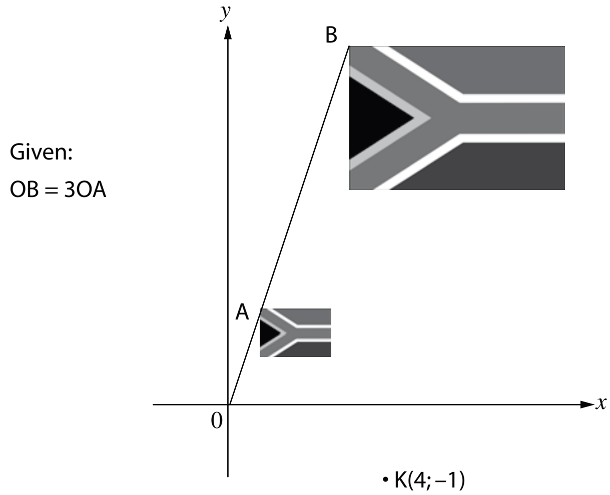
3.2.1



3.2.2



3.2.3

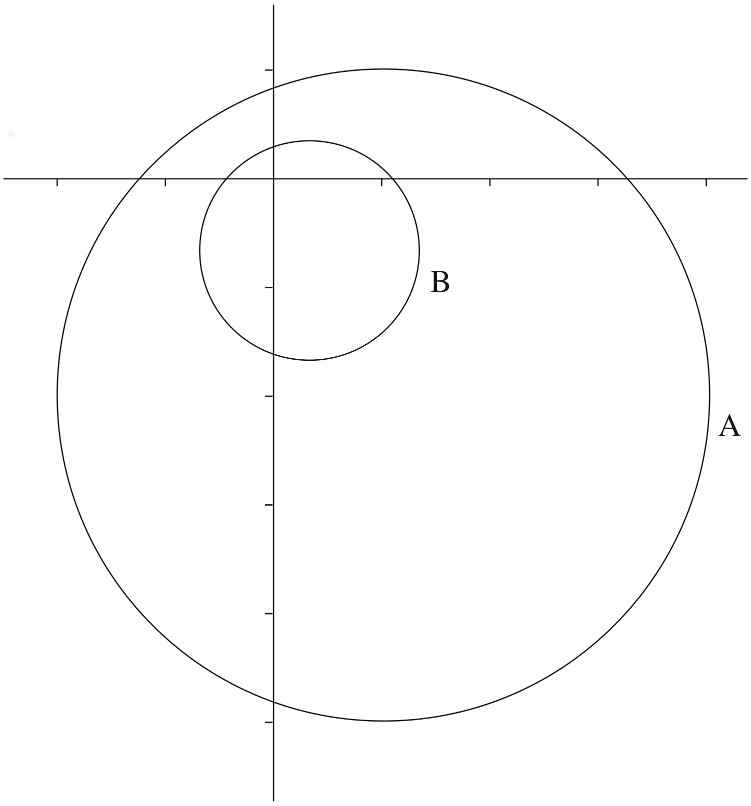


For Enrichment (you will only learn circles in Grade 12):

4. In the diagram, two circles have been drawn. Circle A is an enlargement of circle B by a factor of k.

Circle A: $(x-1)+(y+2)=t^2$ and

Circle B: $(x-a)^2+(y-b)^2=r^2$

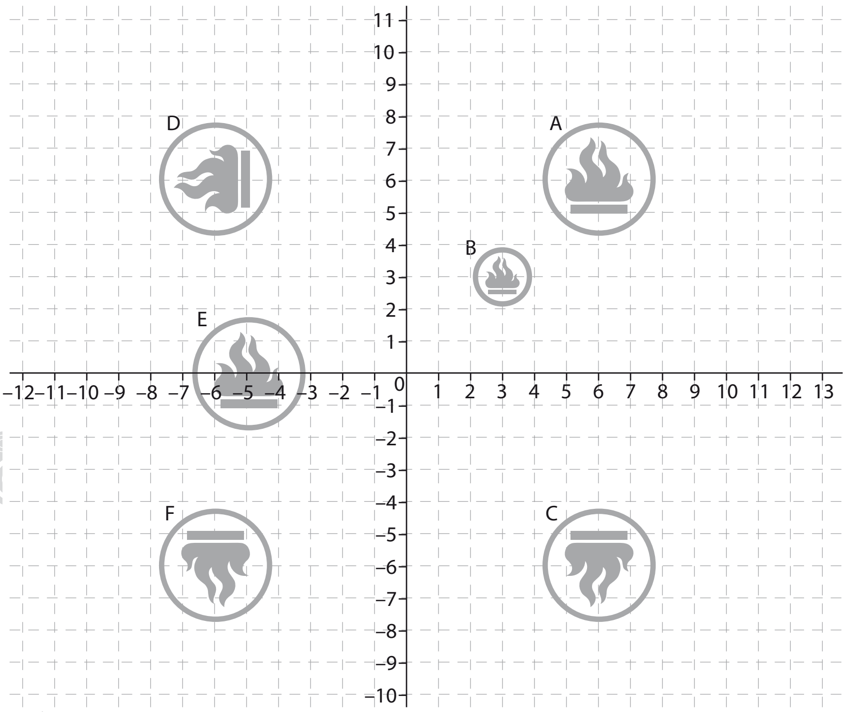


If it is further given that $\frac{\text{Area of circle A}}{\text{Area of circle B}} = 9$, determine

4.3 the values of k, a, and b

4.4 a pair of possible values for r and t.

5. Write a single rule for each of the transformations of the liberty life logo in the form $(x; y) \rightarrow \ldots$



5.1 A to B

5.2 A to C

5.3 A to D

5.4 A to E

5.5 A to F

5.6 C to D

Solutions

1.1 3

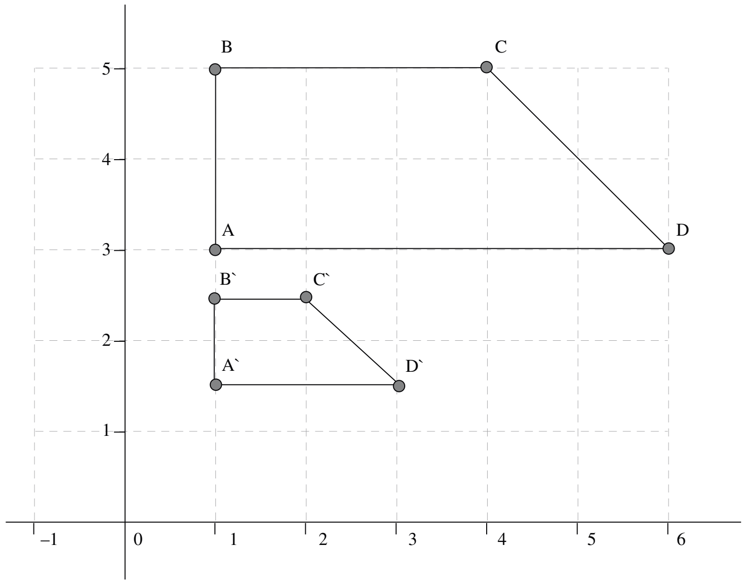
1.2 20

1.3 8

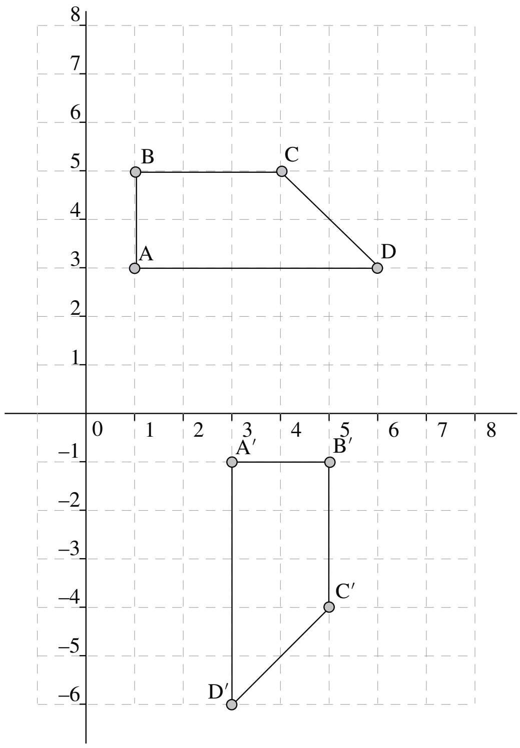
1.4 $(-a; -b)$

1.5 $(b; -a)$

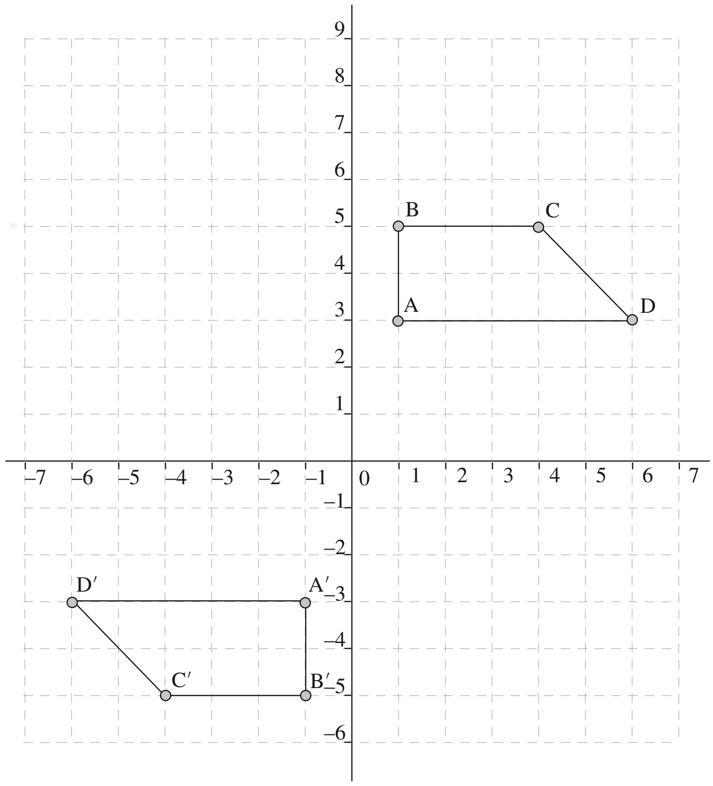
2.1



2.2



2.3



3.1.1 K(1; 4)

3.1.2 Rotation of 90° about the origin in an anticlockwise direction.
$(x; y) \rightarrow (-y; x)$

3.2.1 K(–4; 1)

3.2.2 Rotation of 180° about the origin. $(x; y) \rightarrow (-x; -y)$

3.3.1 $K\left(\frac{4}{3}; -\frac{1}{3}\right)$

3.3.2 Enlargement through the origin by a factor of $\frac{1}{3}$: $(x; y) \rightarrow \left(\frac{x}{3}; \frac{y}{3}\right)$

4.1 We know that $\frac{\text{Area of circle A}}{\text{Area of circle B}} = k^2$ $\quad \therefore k^2 = 9 \quad \therefore k = 3$

Now, centre of circle A is (1; –2) Therefore centre of circle B is $\left(\frac{1}{3}; -\frac{2}{3}\right)$.

Thus $a = \frac{1}{3}$ and $b = -\frac{2}{3}$

4.2 Since the factor of enlargement is 3, the radius of circle A is 3 times the radius of circle B. Therefore $r = 1$ and $t = 3$ are possible values.

5.1 $(x; y) \rightarrow \left(\frac{1}{2}x; \frac{1}{2}y\right)$

5.2 $(x; y) \rightarrow (x; -y)$

5.3 $(x; y) \rightarrow (-y; x)$

5.4 $(x; y) \rightarrow (x - 11; y - 6)$

5.5 $(x; y) \rightarrow (-x; -y)$

5.6 $(x; y) \rightarrow (y; x)$ (reflection about the line $y = x$)