LESSON 11

VOLUMES AND SURFACE AREAS

In this lesson we will

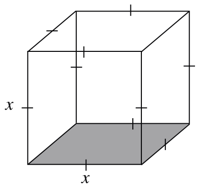
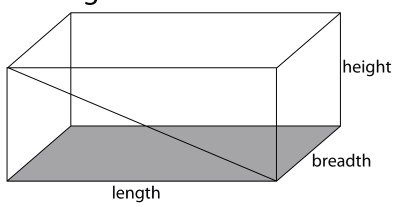
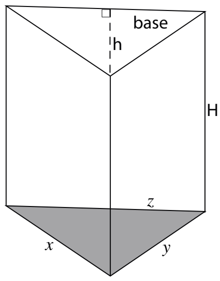
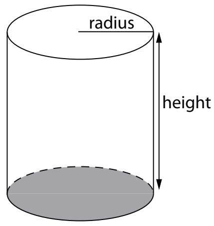
- Review the formula from Grade 10
- Learn the new formulae and apply them to mensuration problems

Firstly, remember to think of **volume** as the amount of liquid or air that a 3-D shape can hold; and **surface area** as the exterior surface of the shape that you could paint.

In Grade 10 you learnt that, in general, to work out

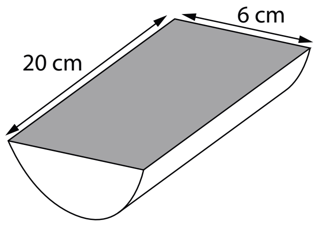
1. **Volume**: we take the area of the base and multiply that by the height
2. **Surface area** we find out the area of each face seperately and then add the answers together to get a total.

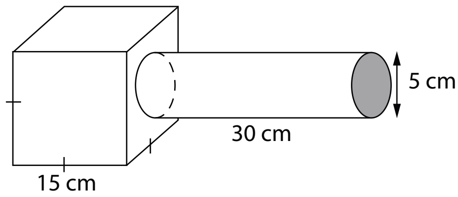
Here is a reminder of formulae from Grade 10:

Shape	**Volume**	**Surface area**
Cube 	$V = \underbrace{(x \times x)}_{\text{base area square}} \times \underbrace{x}_{\text{height}}$ $= x^3$	$SA = 6 \times (x \times x)$ $= 6x^2$
Rectangular Prism 	$V = \underbrace{l \times b}_{\text{base area rectangle}} \times \underbrace{h}_{\text{height}}$	$SA = 2lb + 2bh + 2lh$ $= 2(lb + bh + 1h)$
Triangular Prism 	$V = \underbrace{\left(\frac{1}{2}b \times h\right)}_{\text{area of triangular base}} \times \underbrace{H}_{\text{height}}$	SA = two triangles + 3 rectangles $= 2\left(\frac{1}{2}b \times h\right) + (x + y + z)\,H$ $= b \times h +$ (perimeter $\times H$)
Cylinder 	$V = \underbrace{\pi r^2}_{\text{area of circular base}} \times \underbrace{h}_{\text{height}}$ $= \pi r^2 h$	$SA = 2\pi r^2 +$ curved rectangle $= 2\pi r^2 + 2\pi rh$ $= 2\pi r\,(r + h)$

Example

Here are two quick examples to refresh your skills:

1. 

2. 

Calculate:

a. the volume

b. surface area of these 3-D shapes

Solution

1.a Volume = half volume of cylinder

$= \frac{1}{2}(\pi r^2 h)$

$= \frac{1}{2}(\pi(3)^2(20))$

$= 282,74 \text{ cm}^3$

1.b Surface Area = $2(\frac{1}{2}$ circles) + shaded rectangle + $\frac{1}{2}$ curved rectangle

$= 2(\frac{1}{2}\pi r^2) + (\text{length} \times \text{breadth}) + \frac{1}{2}(2\pi r \times \text{H})$

$= 9\pi + 120 + 60\pi$

$= 336,77 \text{ cm}^2$

Can you see that although surface area is harder to evaluate, the best way is to break it down into the sum of its parts?

2.a Volume = x^3 = Volume of the cube + volume of the cylinder

$= 15^3 + \pi(2,5)^2(30)$

$= 3964,05 \text{ cm}^3$

2.b **Surface Area**= 6 square faces – **circle where handle meets (x) + end shaded circle (y)** + curved rectangle

$= 6x^2 + 2\pi r\text{h}$

$= 6(15)^2 + 2\pi(2,5)(30)$

$= 1821,24 \text{ cm}^2$

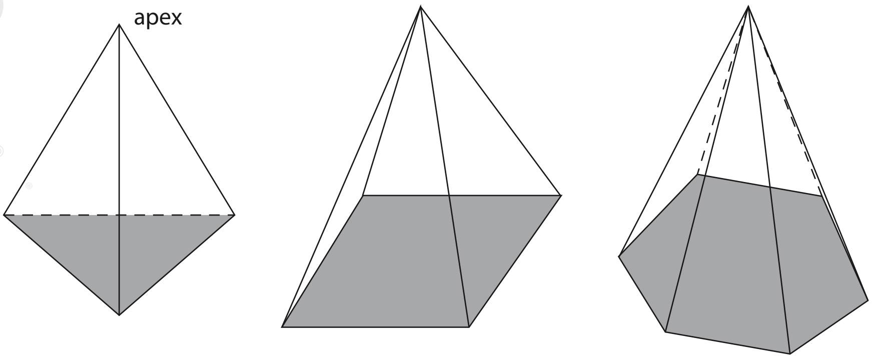
$x = y$, therefore they cancel each other out.

You are now ready to meet the shapes we study in Grade 11. The first family of shapes are called the

Right Pyramids

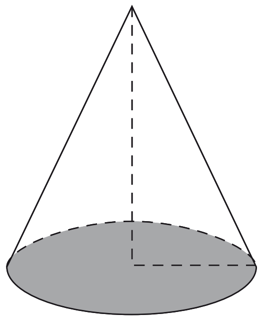
- These are 3-D shapes with a polygon as a base and triangular sides which meet at the top point, called an apex.
- A right pyramid has the apex directly above the middle of the base.
- The pyramid is named according to the shape of the base.
- We study pyramids with regular polygons in their base so that their slanted sides are all congruent triangles.

Triangular Pyramid **Square Pyramid** **Hexagonal Pyramid**



We are also required to study:

Cones – although we group these separately, a cone is actually a pyramid with a circular base

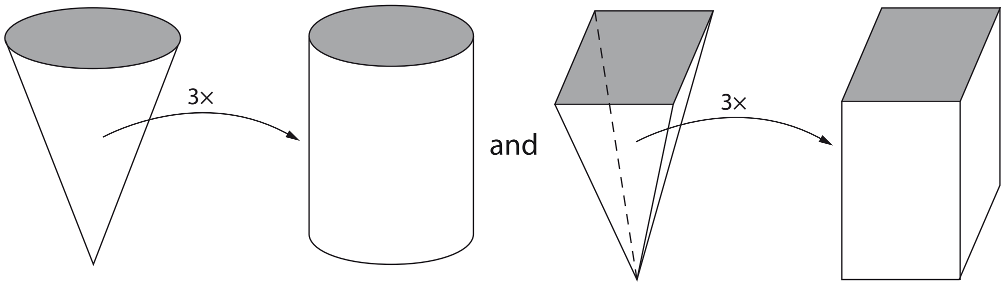


Now that we have introduced these shapes to you, let us look at the formulae for their volume and surface area.

> **The volume** of any pyramid is:
>
> $$V = \frac{1}{3} \times \text{area of the base} \times \text{perpendicular height}$$

Hopefully you will be able to see where the volume formulae come from:

Look at this:



Can you see that by filling the cone three times over we could fill a cylinder provided they had the same radius and height.

Likewise 3 fills of the square based pyramid would fill up the square based prism. Maybe you can get your teacher to help you make models of these shapes and to demonstrate this by using sand or jelly tots.

The Surface Area of any **Pyramid** is:

SA = the sum of the areas of the seperate faces

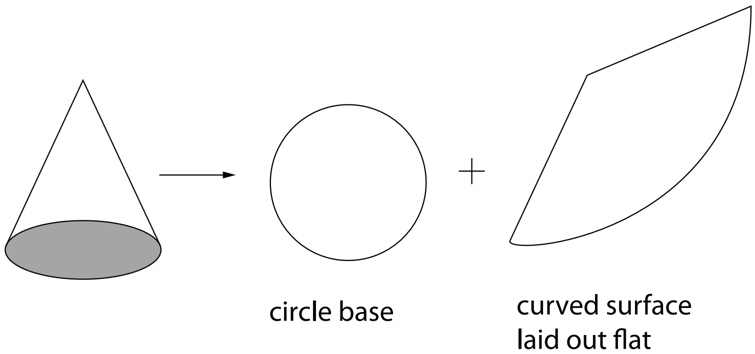
= the area of the base + area of all the congruent triangles.

The Surface Area of a **Cone** is:

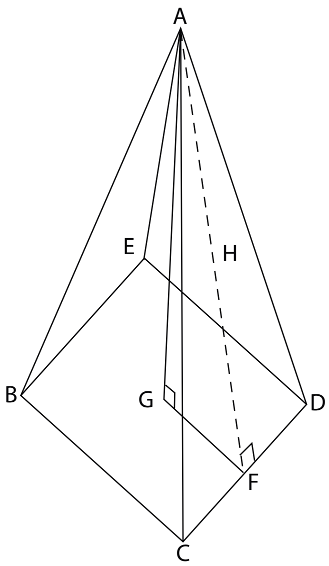
SA = the area of the circular base + area of curved surface

$= \pi r^2 + \pi r \times$ (slant height)

Surface area of a cone:



We now need you to study the next two shapes carefully, so that you are familiar with all the terminology we will be using in the next exercises.



BCDE is the **base** of the pyramid

AG is the **height** of the pyramid

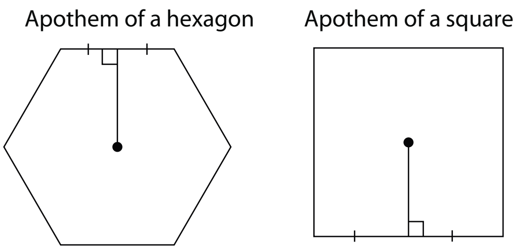
CD is the **base** of $\triangle$ACD

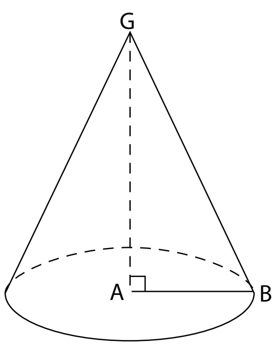
AF is the **slant height** of the pyramid, but the **height** of $\triangle$ACD

$A\hat{F}G = \theta$ is the angle at which the face is slanting

GF is called the **apothem**.

An apothem of a regular polygon is just a line segment drawn from the centre to the midpoint of one of its sides, and it is perpendicular to the side.





AB is the radius of the circular base

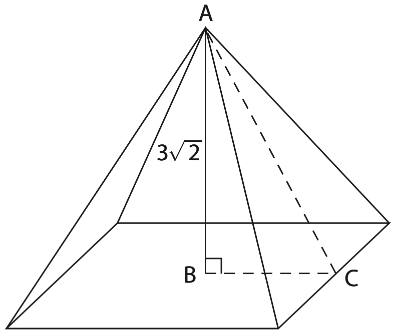
AG is the height of the cone

BG is called the slant height

You are also going to need to know the **Theorem of Pythagoras** which states that for $\triangle ABG$ above: $AB^2 + AG^2 = BG^2$ in order to work out the height, slant height or radius/apothem.

Example 1

Find the volume and surface area of a pyramid with height $3\sqrt{2}$ cm and a square base with sides all 6 cm in length.

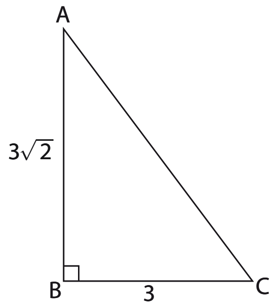


Solution

- Volume $= \frac{1}{3} \times$ area of base $\times h$

$$= \frac{1}{3} \times (6 \times 6) \times 3\sqrt{2}$$
$$= 36\sqrt{2} \text{ cm}^3$$
$$= 50{,}91 \text{ cm}^3$$

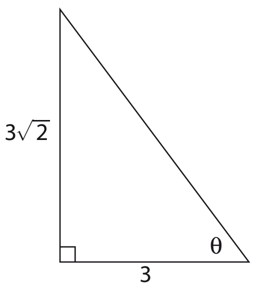
- Surface Area = area of square + area 4 triangle
  We need slant height AC



$$AC^2 = 3^2 + (3\sqrt{2})^2 \text{ (By Pythagoras)}$$
$$AC^2 = 27$$
$$AC = \sqrt{27}$$
$$\therefore SA = 6 \times 6 + 4\left(\frac{1}{2} \times 6 \times \sqrt{27}\right)$$
$$= 98{,}35 \text{ cm}^2$$

Notice that we could work out the angle that the triangles were slanted at, using basic trigonometry.

Since in $\triangle ABC$

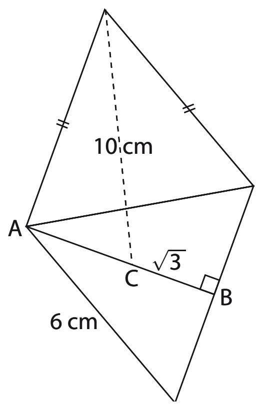


$$\tan\theta = \frac{3\sqrt{2}}{3}$$
$$\theta = 54{,}74°$$

How wonderful is that!

Example 2

Find the volume and surface area of a pyramid with a triangular base with each edge 6 cm and a height of 10 cm.



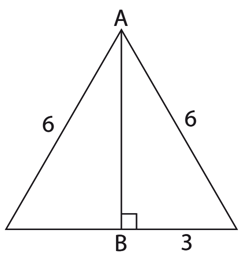
Solution

- Volume $= \frac{1}{3} \times$ area of base $\times h$
  To work out area of base triangle

$$AB^2 = 6^2 - 3^2$$
$$AB = \sqrt{27}$$

$9\sqrt{3}$

$$\therefore \text{Volume} = \frac{1}{3} \times \left(\frac{1}{2} \times 6 \times \sqrt{27}\right) \times 10$$
$$= 30\sqrt{3}$$
$$= 51{,}96 \text{ cm}^3$$



Or to work out the area of the base we could use the area rule from Trigonometry so area = $\frac{1}{2}(6)(6)\sin 60°$ (equilateral △ has int. angle = 60°)

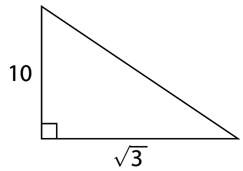
$= 9\sqrt{3}$

- Area base + 3 other △'s

  $9\sqrt{3} + 3\left(\frac{1}{2} \times 6 \times \sqrt{103}\right) = 106{,}93 \text{ cm}^2$

  Slant height = $\sqrt{103}$

  $= 62{,}35 \text{ cm}^3$



Example 3

Find the volume and surface area of a cone with diameter 8 cm and height 8 cm.

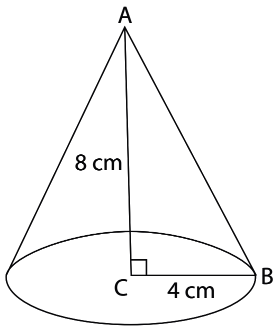
Solution

- Volume = $\frac{1}{3}\pi r^2 h$

  $= \frac{1}{3}\pi(4)^2(8)$

  $= 134{,}04 \text{ cm}^3$

Remember if diameter = 8 cm, we use radius = 4 cm



- Surface Area = area of base circle + area curved surface

  $= \pi r^2 + \pi r$ (slant height)

Now we need to use Pythagoras to get the slant heights:

$s^2 = 8^2 + 4^2$

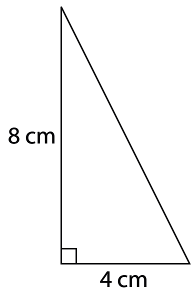
$s^2 = 80$

$s = \sqrt{80}$

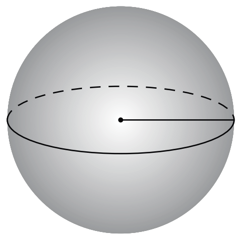
so now

$SA = \pi(4)^2 + \pi(4)(\sqrt{80})$

$= 162{,}66 \text{ cm}^2$



Spheres – this is a body bounded by a surface whose every point is the same distance from a centre point. For example: a soccer ball, a tennis ball. Remember that half a sphere is called a HEMISPHERE



We now just need to give you the formulae for measuring the volume and surface area of a sphere.

For a sphere:
Volume = $\frac{4}{3}\pi r^3$ (where r is the sphere's radius)
Surface Area = $4\pi r^2$

Example 1

An ice cream cone has a radius of 3 cm and a height of 12 cm. A half scoop of ice cream is paced on the cone. If the ice cream melts, will it fit into the cone?

Solution

Solution

Volume cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(3)^2(12)$

$= 36\pi$

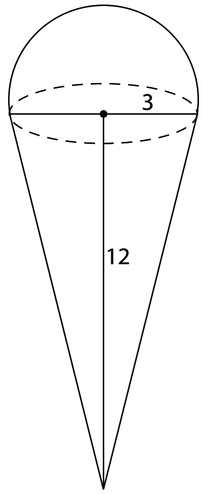
$= 113{,}1 \text{ cm}^3$

Volume scoop $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{1}{2}\left(\frac{4}{3}\pi(3)^3\right)$

$= 18\pi$

$= 56{,}55 \text{ cm}^3$



So yes, it will fit and will actually only half fill the cone.

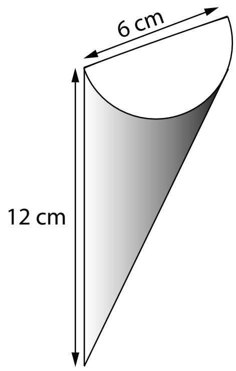
All you now need to do is practise applying these formulae:

Activity

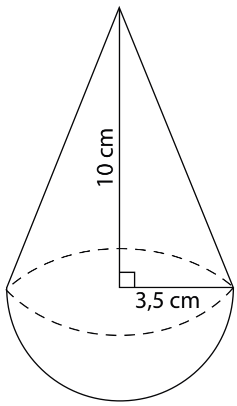
Activity 1

Find the volume of these shapes (correct to 2 decimal digits)

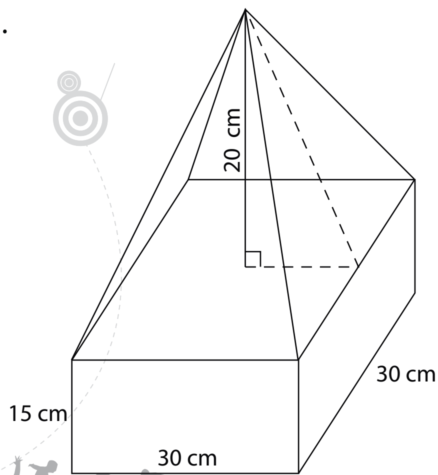
1.



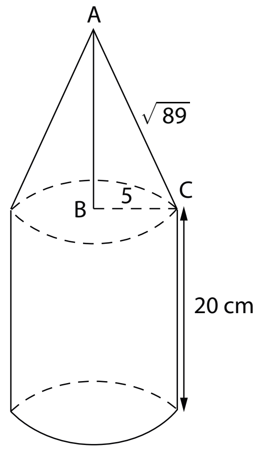
2.



3.



4.



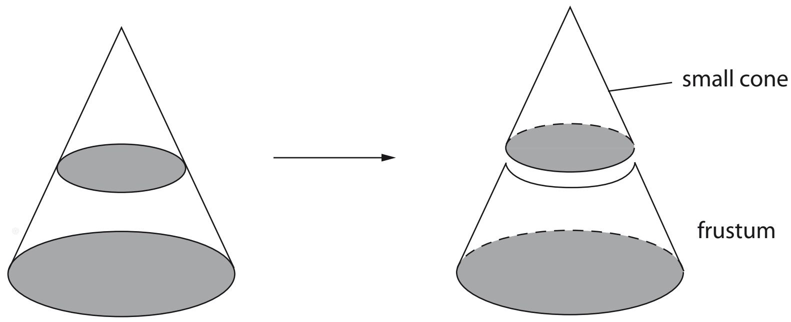
Activity 2

Find the surface area of the shapes 1– 4 in Exercise 1.

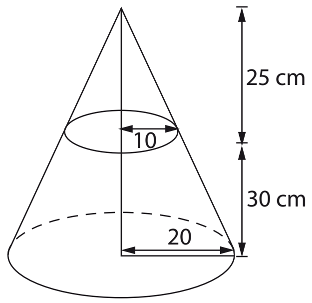
Activity 3

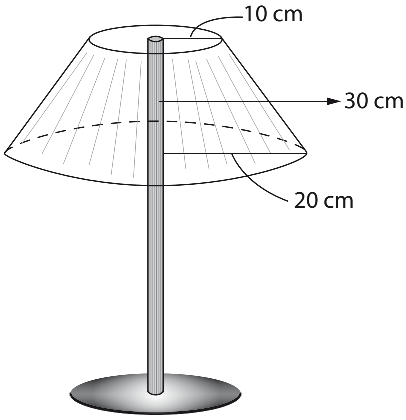
To finish off this module we are going to challenge you to use your knowledge to solve real life problems.

1. If we take a cone and we remove the top of it by making a cut parallel to the circular base we get a small cone and a shape called a **frustum**.



Ayanda wants to make a lamp shade. She first makes a cone with dimensions as shown and then cuts off the top. She covers the frustum with material. How much material does she need?

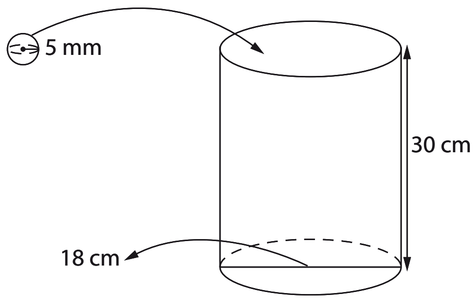




2. Paul wants to put 1 200 steel ball bearings (which are spherical in shape) with a radius of 5 mm, into a cylindrical container which is 30 cm high and 18 cm in diameter.

   Will all the bearings fit into the container?

   Show all your working.

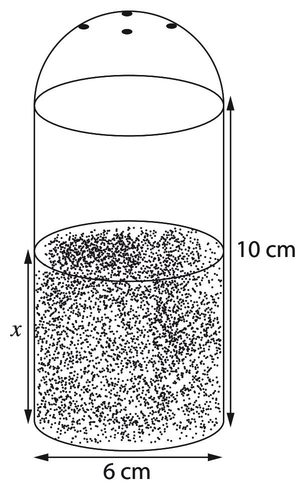


3. Abigail buys a new salt cellar in the shape of a cylinder topped by a hemisphere, as shown below.

   The cylinder has a diameter of 6 cm and a height of 10 cm.

   She pours the salt into the salt cellar, so that it takes up half the total volume of the pot.

   Find the depth of the salt, marked with x in the diagram.



4. The pyramid of the sun of Teotihuacan in mexico is the third largest pyramid in the world behind the great Pyramid of Cholula and the Great Pyramid of Giza.

   It is a right pyramid with a base that is approximately square.

   The length of one side is approximately 223,5 m and the height is about 71,2 m

Calculate:

4.1 The angle at which the slanted face is sloping

4.2 By how much would the volume of material increase if the Pyramid was 1 m Higher/taller?

4.3 The volume of material in a pyramid that has dimensions that are half the original ie: Length of square base = 111,75 m Height = 35,6 m

4.4 The ratio of the pyramid volume calculated in 4.3 above, to the volume of the original pyramid.

Solutions to Activities

Activity 1

1. Volume $= \frac{1}{2}$ (volume of cone)

$= \frac{1}{2}\left(\frac{1}{3}\pi r^2 h\right)$

$= \frac{1}{6}(\pi)(3)^2(12)$

$= 18\pi$

$= 56{,}55 \text{ cm}^3$

2. Volume = volume cone + $\frac{1}{2}$ sphere

$= \frac{1}{3}\pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{1}{3}\pi(3{,}5)^2(10) + \frac{2}{3}\pi(3{,}5)^3$

$= \frac{245}{6}\pi + \frac{343}{12}\pi$

$= \frac{833}{12}\pi$

$= 218{,}08 \text{ cm}^3$

3. Volume = volume prism + volume pyramid

$= 30 \times 30 \times 15 + \left(\frac{1}{3} \times 30 \times 30 \times 20\right)$

$= 13\,500 + 6\,000$

$= 19\,500 \text{ cm}^3$

4. Volume = volume cylinder + volume cone

$= \pi r^2 h + \frac{1}{3}\pi r^2 h$

$= \pi(5)^2(20) + \frac{1}{3}\pi\, 5^2(AB)$

Need to find AB : $AB^2 = AC^2 - BC^2$

$AB^2 = 89 - 25$

$AB = 8$

$= \pi(5)^2\, 20 + \frac{1}{3}\pi(5)^2 8$

$= 500\pi + \frac{200\pi}{3}$

$= \frac{1700}{3}\pi$

$= 1780{,}24 \text{ cm}^3$

Activity 2

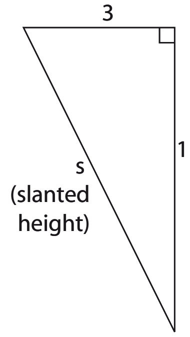
1. Surface area $= \frac{1}{2}$ circle $+ \frac{1}{2}$ curved area + triangle

$= \frac{1}{2}(\pi r^2) + \frac{1}{2}(\pi r(s)) + \frac{1}{2}$ base $\times$ h

$= \frac{1}{2}\pi(3)^2 + \frac{1}{2}\pi(3)(\sqrt{12^2 + 3^2}) + \frac{1}{2}(6)(12)$

$= \frac{9}{2}\pi + 58,29 + 36$

$= 108,43$ cm^2



2. Surface area $= \frac{1}{2}(4\pi r^2)$ + curved area

$= \frac{1}{2}(4\pi(3,5)^2) + \pi r(s)$

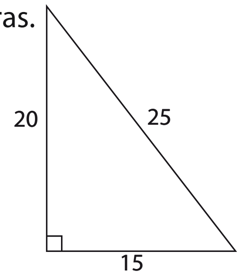
$= \frac{49}{2}\pi + \pi(3,5)\left(\sqrt{100 + (3,5)^2}\right)$

$= 193,47$ cm^2

3. Surface area = area of 5 surfaces of prism + area 4 triangles

We need to know the height of each triangle, so we calculate it by the theorem of Pythagoras.

Note: We can get the **apothem** as half the side length



So in the triangle $S^2 = 20^2 + 15^2$

$S^2 = 400 + 225$

$S^2 = 625$

$S = 25$

So surface area becomes $= (30 \times 30) + 4(30 \times 15) + 4(\frac{1}{2} \times 30 \times 25)$

$= 900 + 1800 + 1500$

$= 4\,200$cm^2

4. Surface area = base circle + curved bit + curved cone area

$= \pi(5)^2 + 2\pi(5)(20) + \pi(5)(\sqrt{89})$

$= 855,05$ cm^2

Activity 3

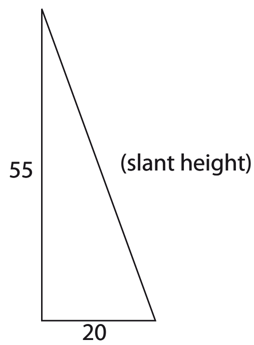
We will get the surface area of the frustum

1. We work out the surface area of the whole cone and then subtract the surface area of the small one. We do not need the circular base.

Big Cone SA $= \pi r s$

$= \pi(20)\sqrt{(55)^2 + (20)^2}$

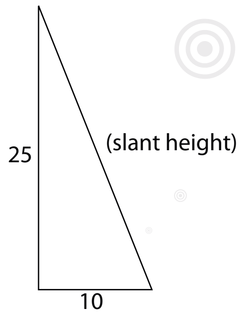
$= 3\,677,14$ cm^2



Little Cone SA $= \pi r s$

$$= \pi(10)\sqrt{(25)^2 + (10)^2}$$

$$= 845{,}90 \text{ cm}^2$$



$\therefore$ Frustum/ lamp shade needs 3 677, 14 – 845, 90

$$= 2\,831{,}24 \text{ cm}^2 \text{ of material}$$

2. We are going to work in cm so first we change

$5 \text{ mm} \rightarrow 0{,}5 \text{ cm} = \frac{1}{2} \text{ cm}$

Each ball bearing has volume $= \frac{4}{3}\pi(r)^3 = \frac{4}{3}\pi(\frac{1}{2})^3 = \frac{4}{3}\pi(\frac{1}{8})$

$$= \frac{4}{3} \cdot \frac{1}{8}\pi$$

$$= \frac{1}{6}\pi = \frac{\pi}{6} \cong 0{,}52\text{cm}^3$$

Cylinder has volume $= \pi r^2 h$

$= \pi(9)^2(30)$

$= 2\,430\pi$

$= 7\,634{,}07 \text{ cm}^3$

Now 1 200 bearings will have volume $1200 \times \frac{\pi}{6} = 200\pi \cong 628{,}32 \text{ cm}^3$ so they will be able to fit in with lots of room to spare.

3. Volume of salt cellar $= \pi r^2 h + \frac{1}{2} \times \left(\frac{4}{3}\pi r^3\right)$

$$= \pi(3)^2(10) + \frac{1}{2} \times \frac{4}{3} \times \pi(3)^3$$

$$= 108\pi \cong 339{,}29 \text{ cm}^3$$

So half the total volume $= 54\pi$ or $169{,}65 \text{ cm}^3$

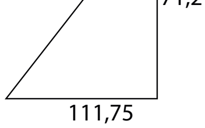
We want to find x so that $\pi r^2 x = 169{,}65$

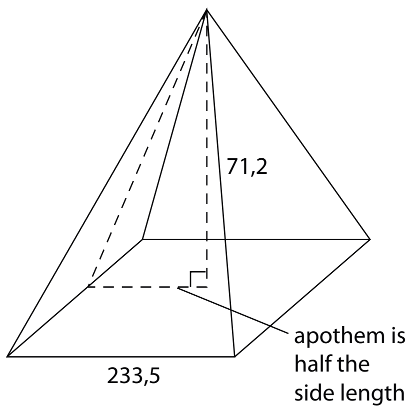
$$\pi(3)^2 x = 169{,}65$$

$$x = 6 \text{ cm}$$

So the salt will be 6 cm deep.

4.1





Now $\tan\theta = \frac{71{,}2}{111{,}75}$

$$\theta = 32{,}5°$$

4.2 Current volume $= \frac{1}{3}$ base area – h

$= \frac{1}{3}(223,5)^2 \times 72,2$

$= 1185533,4\ \text{m}^3$

If increased height by 1 m $= \frac{1}{3}(223,5)^2 \times 72,2$

$= 1202184,15\ \text{m}^3$

$\therefore$ an extra 16650,75 m^3

4.3 Volume $= \frac{1}{3}(111,75)^2 \times 35,6$

$= 148191,675\ \text{m}^3$

4.4 $\frac{148191,675}{1185533,4} = 0,125 = \frac{1}{8}$

$\therefore$ ratio 1: 8

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