TRIGONOMETRY

Special Angles: 30°; 60° and 45°

Overview

In this lesson you will:

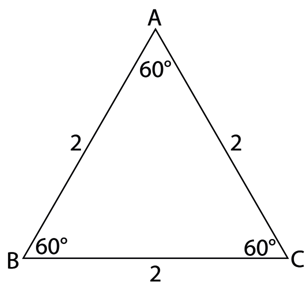
- Use an equilateral triangle to define the sine; cosine and tangent ratios of 30° and 60°
- Use an isosceles triangle to define the sine; cosine and tangent ratios of 45°
- Simplify trigonometric expressions.

Draw an equilateral $\triangle ABC$ with side 2 units.

Definition: An equilateral $\triangle$ has all sides equal and all 3 angles equal 60° each.

$\therefore$ In the case alongside: $AB = BC = AC = 2$ units

$\hat{A} = \hat{B} = \hat{C} = 60°$



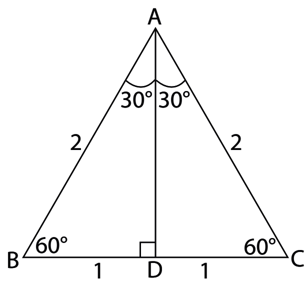
Drop a perpendicular from A onto the opposite side BC.

We say $AD \perp BC$.

The perpendicular bisects $\hat{A}$ and bisects the opposite side BC.

$\therefore \hat{A}_1 = \hat{A}_2 = 30°$ and

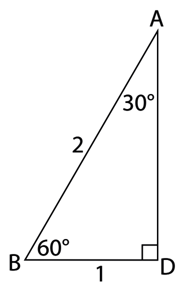
$\therefore BD = DC = 1$ unit



Take a look at $\triangle ABD$.

How would you calculate the length of BD? Why?

Since we have the hypotenuse, AB = 2 units, and BD = 1 unit, then the 3rd side of a **right-angled triangle** is calculated using **Pythagoras**.



Using Pythagoras:

$AD^2 + BD^2 = AB^2$

$AD^2 = (2)^2 - (1)^2$

$AD^2 = 3$

$AD = \sqrt{3}$

We can now determine the trigonometric ratios (trig ratios) of 60° and 30° using the three definitions:

$$\text{sine } \theta = \sin\theta = \frac{\text{side OPPOSITE the angle}}{\text{hypotenuse}}$$

$$\text{cosine } \theta = \cos\theta = \frac{\text{side ADJACENT to the angle}}{\text{hypotenuse}}$$

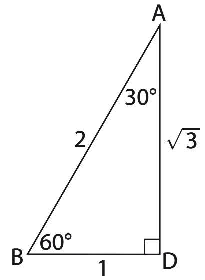
$$\text{tangent } \theta = \tan\theta = \frac{\text{side OPPOSITE the angle}}{\text{side ADJACENT to the angle}}$$

In short we can use, SOHCAHTOA, to remind us of the 3 fractions:

$\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{BD}{AB} = \frac{1}{2}$

$\tan 60° = \frac{AD}{BD} = \frac{\sqrt{3}}{1}$



$\sin 30° = \frac{BD}{AB} = \frac{1}{2}$

$\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{BD}{AD} = \frac{1}{\sqrt{3}}$ $(= \frac{\sqrt{3}}{3}$ when simplified using a calculator)

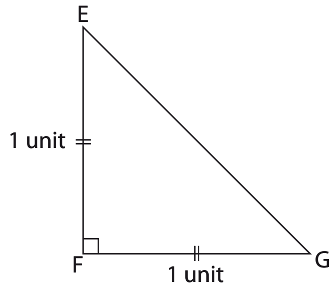
60° and 30° are special angles.

There is one more **special angle.**

Draw **isosceles** right angled $\triangle EFG$ of side 1 unit.

Definition: Isosceles means two sides are of equal length.

$\therefore$ In the diagram alongside EF = FG = 1 unit.



How do you calculate length GE?

GE is the hypotenuse as it is the side opposite the right angle.

$\therefore (GE)^2 = (EF)^2 + (FG)^2$ (Pythagoras)

$= 1^2 + 1^2$

$= 2$

$\therefore GE = \sqrt{2}$

What is the true size of $\hat{E}$ and $\hat{G}$?

Since $\hat{E}$ and $\hat{G}$ are opposite equal sides in an isosceles $\triangle$, $\hat{E} = \hat{G} = 45°$

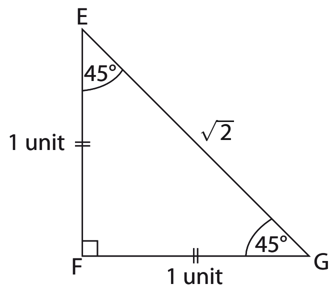
$\therefore$ We can determine the trig ratios of 45°:

$\sin 45° = \frac{EF}{EG} = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{FG}{EG} = \frac{1}{\sqrt{2}}$

$\frac{\sqrt{2}}{2}$ when simplified using a calculator.

$\tan 45° = \frac{EF}{FG} = \frac{1}{1} = 1$



Example

Example

1. Simplify the following without a calculator:

1.1.1 $\frac{\sin 60°}{\cos 60°}$

1.1.2 $\cos^2 30° + \sin^2 30°$

1.1.3 $\cos^2 45° - \sin^2 45°$

2. $(\cos 45° + \sin 45°)^2 \neq \cos^2 45° + \sin^2 45°$

Explain, using special ratios, why the above statement is true. Correct the right hand side so that "≠" becomes "=".

Solution

Solution

1.1.1 $\frac{\sin 60°}{\cos 60°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

1.1.2 $\cos^2 30° + \sin^2 30° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

1.1.3 $\cos^2 45° - \sin^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0$

2. LHS $(\cos 45° + \sin 45°)^2$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2$

$= \left(\frac{2}{\sqrt{2}}\right)^2$

$= \frac{4}{2}$

$= 2$

RHS $\cos^2 45° + \sin^2 45°$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$

$\therefore (\cos 45° + \sin 45°)^2 \neq \cos^2 45° + \sin^2 45°$

The RHS has a missing term: 2 cos 45° sin 45°, which is the 'middle' term of the bracket squared.

$\therefore (\cos 45° + \sin 45°)^2 = \cos^2 45° + 2 \cos 45° \sin 45° + \sin^2 45°$

Activity

Activity 1

1. Simplify without a calculator:

1.1 $\frac{\cos 30°}{\sin 30°}$

1.2 $\sin^2 60° + \cos^2 60°$

1.3 $(\cos 60° + \sin 60°)^2$

2.1 Is $\cos^2 60° + \sin^2 60° = \cos^2 45° + \sin^2 45°$?

2.2 Is $(\cos 60° + \sin 60°)^2 = (\cos 45° + \sin 45°)^2$?

2.3 Comment on what you notice in Question 2.1 and 2.2

....

....

....

Trigonometric Identities:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$

$\sin^2\theta + \cos^2\theta = 1$

Overview

In this lesson you will:

- Use the trigonometric ratios of sine; cosine and tangent of 30°; 45° and 60° to verify the identities.
- Establish "Thinking Tips" on how to use the above identities.
- Use the above identities to simplify trigonometric expressions on the left hand side (LHS) and right hand side (RHS) of an identity.
- Use the above identities to prove an identity (ie show that the LHS and RHS of the identity can be written in the same way).

Simplifying $\frac{\sin 60^\circ}{\cos 60^\circ}$ gives $\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

but tan 60° also equals $\sqrt{3}$.

$\therefore \boxed{\frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ}$

Simplify $\frac{\sin 30^\circ}{\cos 30^\circ}$ and $\frac{\sin 45^\circ}{\cos 45^\circ}$. What do you notice?

$\frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}$ and tan 30° also equals $\frac{1}{\sqrt{3}}$

$\frac{\sin 45^\circ}{\cos 45^\circ} = 1$ and tan 45° also equals 1.

$\therefore$ We say that $\boxed{\frac{\sin\theta}{\cos\theta} = \tan\theta}$ and for any angle θ, the quotient of $\frac{\sin\theta}{\cos\theta}$ will always equal $\tan\theta$.

This relationship is known as an **identity**.

$\therefore \tan 20^\circ = \frac{\sin 20^\circ}{\cos 20^\circ}$; $\tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ}$; $\tan 65^\circ = \frac{\sin 65^\circ}{\cos 65^\circ}$; etc

Earlier, we simplified $\sin^2 30^\circ + \cos^2 30^\circ$ and we got 1.

Since the sum of the squares of $\sin\theta$ and $\cos\theta$ gives us 1, we form another identity $\sin^2\theta + \cos^2\theta = 1$

From this identity another two versions emerge:

If $\sin^2\theta + \cos^2\theta = 1$

then $\sin^2\theta = 1 - \cos^2\theta$

Also, if $\sin^2\theta + \cos^2\theta = 1$

then $\cos^2\theta = 1 - \sin^2\theta$

$\therefore \cos\theta = \pm\sqrt{1 - \sin^2\theta}$

Summary of identities

Quotient Identity : $\tan\theta = \sin\theta/\cos\theta$, $\cos\theta \neq 0$
Square Identity : $\sin^2\theta + \cos^2\theta = 1$

In terms of sin: $\sin^2\theta = 1 - \cos^2\theta$

In terms of cos: $\cos^2\theta = 1 - \sin^2\theta$

Many more identities exist. We will show that other identities are true by proving that the expression on the left hand side (LHS) equals the expression on the right hand side (RHS).We use the above two identities to prove other identities.

Example

Prove that

1. $\tan\theta + \frac{\cos\theta}{\sin\theta - 1} = -\frac{1}{\cos\theta}$

2. $\frac{\tan\theta\cdot\cos\theta\cdot\sin\theta}{1-\cos^2\theta} = \sin^2\theta + \cos^2\theta$

Solution

1. LHS: $\tan\theta + \frac{\cos\theta}{\sin\theta - 1}$

$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta - 1}$

$= \frac{\sin\theta(\sin\theta - 1) + \cos\theta\cdot\cos\theta}{\cos\theta(\sin\theta - 1)}$

$= \frac{\sin^2\theta - \sin\theta + \cos^2\theta}{\cos\theta(\sin\theta - 1)}$

$= \frac{1 - \sin\theta}{\cos\theta\,(\sin\theta - 1)}$

$= \frac{-1(\sin\theta - 1)}{\cos\theta(\sin\theta - 1)}$

$= -\frac{1}{\cos\theta}$

- Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we replace $\tan\theta$ with its fraction form.
- Add the 2 fractions, using common denominator of $\cos\theta(\sin\theta - 1)$.
- Look carefully at the 2 terms in the numerator: $\sin^2\theta + \cos^2\theta$. These 2 terms add up to 1.
- Terms in the numerator have a different sign. Take out a negative one, factorise and cancel.

RHS: $\frac{-1}{\cos\theta}$ $\therefore$ LHS = RHS $\therefore$ $\tan\theta + \frac{\cos\theta}{\sin\theta - 1} = -\frac{1}{\cos\theta}$

2. LHS: $\frac{\tan\theta\cdot\cos\theta\cdot\sin\theta}{1-\cos^2\theta}$

$= \frac{\frac{\sin\theta}{\cos\theta}\cdot\cos\theta\cdot\sin\theta}{1-\cos^2\theta}$

$= \frac{\sin^2\theta}{\sin^2\theta}$

$= 1$

RHS: $\sin^2\theta + \cos^2\theta = 1$

$\therefore$ LHS = RHS

- $\tan\theta$ is replaced with its fraction form.
- Denominator is a square identity: $1 - \cos^2\theta = \sin^2\theta$
- Divide
- Square Identity
  $\therefore \frac{\tan\theta\cdot\cos\theta\cdot\sin\theta}{1-\cos^2\theta} = \sin^2\theta + \cos^2\theta$

Activity 2

1. Show that $\frac{\tan\theta}{1+\tan^2\theta}$ can be written as $\sin\theta\cdot\cos\theta$

2. Prove that $\frac{\tan\theta\cdot\sqrt{1-\sin^2\theta}}{\sin\theta} = \sin 30° + \cos 60°$, without using a caluculator.

3. If $k = \cos A + \sin A$ and $p = \sin A - \cos A$, determine $k^2 + p^2$ in simplest form.

4. Prove that:

4.1 $\frac{\cos\beta}{\sin\beta} + \tan\beta = (\sin\beta\cdot\cos\beta)^{-1}$

4.2 $\frac{3\cos^2 x - 1 + 3\sin^2 x}{2 - 2\sin^2 x} = \frac{1}{\cos^2 x}$

4.3 $\frac{1}{1+\cos\beta} + \frac{1}{1-\cos\beta} = \frac{2}{\sin^2\beta}$

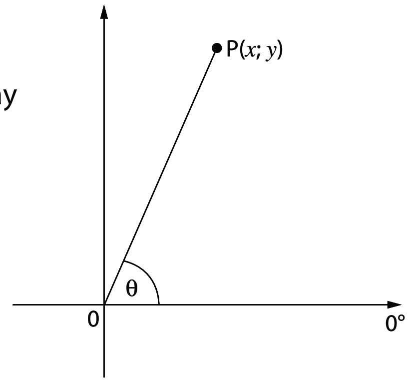
Angles in the Cartesian Plane

Overview

In this lesson you will:

- Define an angle in the Cartesian Plane.
- Define the words: initial ray; terminal ray; anticlockwise; quadrant angles.
- Use x ; y and r to define the sine; cosine and tangent ratios in terms of the co-ordinates of a point and the radius of a circle.

Notation: An angle can be measured in the Cartesian Plane where the initial ray is at 0° on the positive side of the x-axis and the terminal ray rotates in an anti-clockwise direction.



Example: OP, the terminal ray rotates from 0° to its terminal position, θ° anticlockwise.

Vocabulary: If θ is rotated into each quadrant then the

Terminal Ray in Quadrant I forms acute angles.

i.e. $0° < \theta < 90°$

Terminal Ray in Quadrant II forms obtuse angles

i.e. $90° < \theta < 180°$

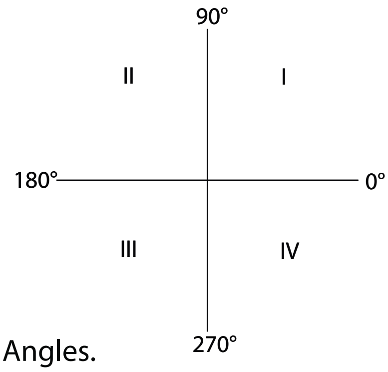
Terminal Ray in Quadrant III forms reflex angles

$180° < \theta < 270°$

Terminal Ray in Quadrant IV forms reflex angles

$270° < \theta < 360°$

Terminal Rays at 0°; 90°; 180°; 270° form Quadrant Angles.



Note:

1. If OP rotates continuously then the path traced out by point P is that of a circle where OP is the radius of the circle. Let's take OP = r (radius = r)

   The point P(x; y) and the radius are related through the following equation:

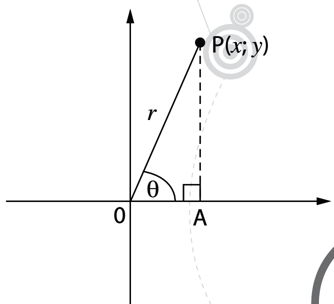
   $x^2 + y^2 = r^2$

   Let OP rotate θ° so that $\theta \in (0° ; 90°)$

2. Draw a perpendicular from P onto the x-axis. This creates right angled △OPA. What is distance OA? PA?

   OA = x units since $x_P = x_A$

   AP = y units



3. What are the 3 trig ratios using △OPA?

   $\sin\theta = \frac{AP}{OP} = \frac{y}{r}$ $\qquad \cos\theta = \frac{OA}{OP} = \frac{x}{r}$

   $\tan\theta = \frac{AP}{OA} = \frac{y}{x}$

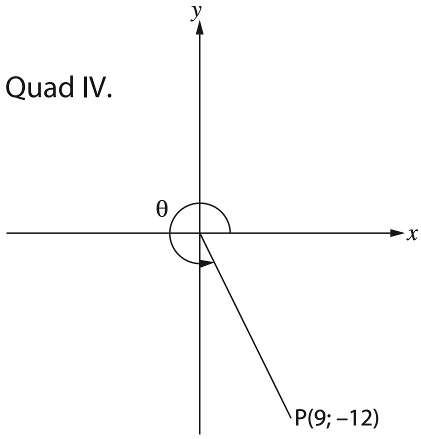
4. Terminal ray OP can rotate to any other quadrant, so if we have the coordinates of P we will be

able to express the 3 trig ratios in terms of the coordinates and the radius.

5. In ANY quadrant: $\sin\theta = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$; $\tan\theta = \frac{y}{x}$

Example

Example

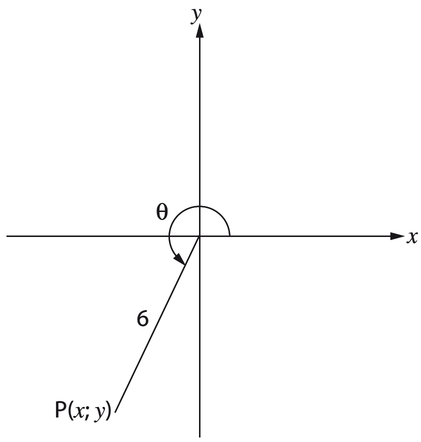


1. P(9; –12) is a point on OP. OP rotates θ to Quad IV.

1.1 Calculate the length of OP.

1.2 Determine the value of the 3 trig ratios, without a calculator.

1.3 Prove $\tan\theta = \frac{\sin\theta}{\cos\theta}$, using the 3 trig ratios.



2. P(x; y) is a point on the terminal ray of angle θ. 180° < θ < 270°.

OP = 6 units.

2.1 If $\cos\theta = \frac{-1}{3}$, determine the value of x.

2.2 Hence calculate the value of y. Leave answer in simplified surd form.

3. 5 sin A – 3 = 0 and 90° < A < 180°,

3.1 In which quadrant will the terminal ray of $\hat{A}$ lie in?

3.2 Determine $\cos^2 A$ without a calculator.

4. If $4\tan\alpha + 5 = 0$ and $\alpha \in (90°; 180°)$, evaluate without the use of a calculator $\sqrt{41}\cos\alpha$.

Solution

Solution

Refer to the diagrams above

1.1 Relationship between x; y and r is $x^2 + y^2 = r^2$ … ①

$\therefore$ substitute $x = 9$ and $y = -12$ into ①

$(9)^2 + (-12)^2 = r^2$

$225 = r^2$

$15 = r$ $(r > 0)$

OP is the radius $\therefore$ OP is 15 units.

Note: These ratios can be simplified further.

1.2 $\sin\theta = \frac{y}{r} = \frac{-12}{15}$; $\cos\theta = \frac{x}{r} = \frac{9}{15}$; $\tan\theta = \frac{y}{x} = \frac{-12}{9}$

1.3 LHS: $\tan\theta = \frac{-12}{9}$

RHS: $\frac{\sin\theta}{\cos\theta} = \frac{\frac{-12}{15}}{\frac{9}{15}} = \frac{-12}{15} \times \frac{15}{9} = \frac{-12}{9} \therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$

2.1 If OP = 6 units, then $r = 6$

If $\cos\theta = \frac{-1}{3}$, that means $\cos\theta = \frac{x}{r}$ and $r = 6$ (given)

$\therefore \cos\theta = \frac{-1}{3} = \frac{-2}{6}$

$\therefore x = -2$

2.2 $x^2 + y^2 = r^2$ sub $r = 6$ and $x = -2$

$\therefore (-2)^2 + (y)^2 = (6)^2$

$y^2 = 32$

$y = \pm\sqrt{32}$

As P lies in Quad III, y_p is negative $\therefore y_p = -4\sqrt{2}$

3.1 $90° < A < 180°$ means that $\hat{A}$ lies in Quadrant II, i.e. $\hat{A}$ is obtuse

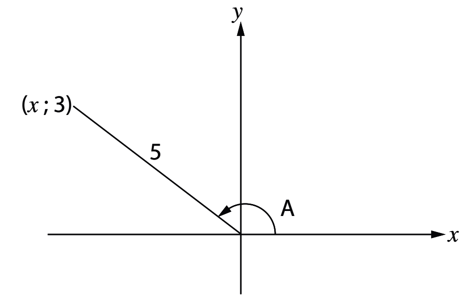
3.2 With the above information, we can draw the terminal ray of $\hat{A}$ in Quad II and fill in that the radius = 5 and a y-value = 3, because $5 \sin A - 3 = 0$

$\therefore 5 \sin A = 3 \therefore \sin A = \frac{3}{5} = \frac{y}{r}$

We can calculate the x-value by using

$x^2 + y^2 = r^2 \quad \therefore x^2 = 25 - 9 = 16$

$\therefore x = \pm 4$



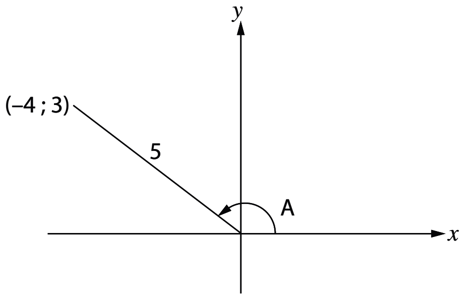
But terminal ray lies in Quad II $\therefore x$ is negative

$\therefore x = -4$

$\cos^2 A = \left(\frac{x}{r}\right)^2$

$= \left(-\frac{4}{5}\right)^2$

$= \frac{16}{25}$



4. $\tan\alpha = \frac{-5}{4}$ and $\alpha \in (90°; 180°)$

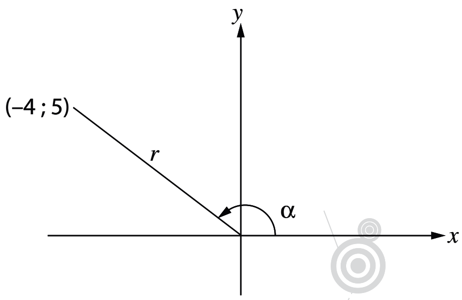
$x^2 + y^2 = r^2 \quad \therefore (-4)^2 + (5)^2 = r^2$

$16 + 25 = r^2$

$\sqrt{41} = r$

$\therefore \sqrt{41}\cos\alpha = \sqrt{41} \cdot \left(\frac{x}{y}\right)$

$= \sqrt{41} \cdot \left(\frac{-4}{\sqrt{41}}\right) = -4$



Note: $\tan\alpha = \frac{-5}{4} = \frac{5}{-4} = -\frac{5}{4}$

All mean the same!

Since the angle is in Quad II, the x-value is negative and $\therefore \tan\alpha = \frac{5}{-4}$ is the fraction which we will use in order to make sure that the x-value is negative and the y-value is positive.

Reduction formulae

Overview

In this lesson you will:

- Investigate the relationship between the trigonometric ratios of **angles larger than 90°** and the trigonometric ratios of **acute angles.**
- Use a calculator to fill in tables and make conclusions from the observations.
- Define the trigonometric ratios for angles in quadrant II; III; IV and angles larger than 360°.
- Establish a "4- question" approach to reducing a trigonometric ratio as an acute angle.
- Simplify expressions and prove identities using reduction formulae.
- Establish the "Co-Co" Rule for Co-functions.
- Define a NEGATIVE ANGLE and CO-TERMINAL ANGLES.

1. What is the connection between 150° and 30°?

   150° is the difference between 180° and 30°, ie 150° = 180° – 30°

   150° has its terminal ray in Quad II.

ALL ANGLES IN QUAD II can be written as 180° – ACUTE ANGLE

For example:	120° = 180° – 60°	170° = 180° – 10°
	135° = 180° – 45°	100° = 180° – 80°, etc.

In general, we say Quad II angle = 180° – θ, where θ is the ACUTE angle.

2. 240° is the sum of 180° and 60° i.e. 180° + 60° = 240°

   240° has its terminal ray in Quad III

All angles in Quad III can be written as 180° + ACUTE ANGLE

For example:	210° = 180° + 30°	250° = 180° + 70°
	225° = 180° + 45°	200° = 180° + 20°, etc.

In general, we say Quad III angle = 180° + θ, where θ is the ACUTE angle.

3. What is the connection between 315° and 45°?

   315° is the difference between 360° and 45°, ie 315° = 360° – 45°

   315° has its terminal ray in Quad IV.

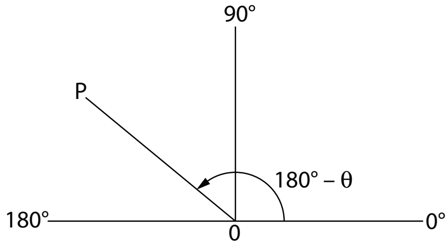
ALL ANGLES IN QUAD IV can be written as 360° – ACUTE ANGLE

For example:	300° = 360° – 60°	280° = 360° – 80°
	330° = 360° – 30°	350° = 360° – 10°, etc.

In general, we say Quad IV angle = 360° – θ, where θ is the ACUTE angle.

Signs and Formulae (θ represents acute angle)

QUAD II

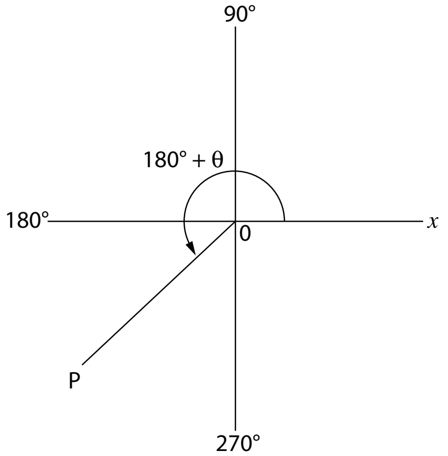


Terminal ray OP is in Quad II

$\therefore$ obtuse angle is $180° - \theta$

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

QUAD III

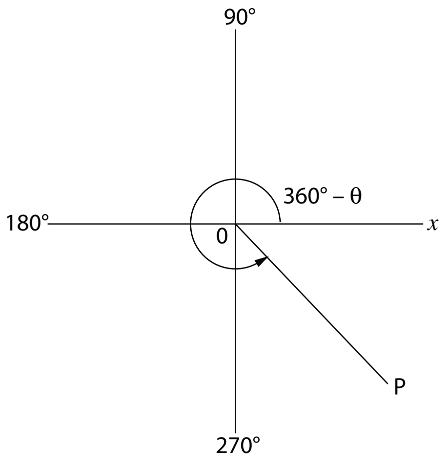


Terminal ray OP is in Quad III

$\therefore$ reflex angle $X\hat{O}P = 180° + \theta$

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$

QUAD IV



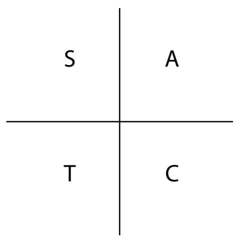
Terminal ray OP is in Quad IV

$\therefore$ reflex angle $X\hat{O}P = 360° - \theta$

$\cos(360° - \theta) = \cos\theta$
$\sin(360° - \theta) = -\sin\theta$
$\tan(360° - \theta) = -\tan\theta$

The formulae in the 3 boxes above are known as **reduction formulae**. We have been able to write the trig ratio of a large angle in terms of a trig ratio of an acute angle.

To help remembering the signs: remember the ratio which is positive in each quadrant



A = All, since all 3 trig ratios are positive in the first quadrant.

Trig ratios of angles larger than 360°

$\sin 780° = \sin(2 \times 360° + 60°) = \sin 60°$

$\cos 1\,200° = \cos(3 \times 360° + 120°) = \cos 120°$

$\sin 600° = \sin(360° + 240°) = \sin 240°$

$\tan 1\,380° = \tan(3 \times 360° + 300°) = \tan 300°$

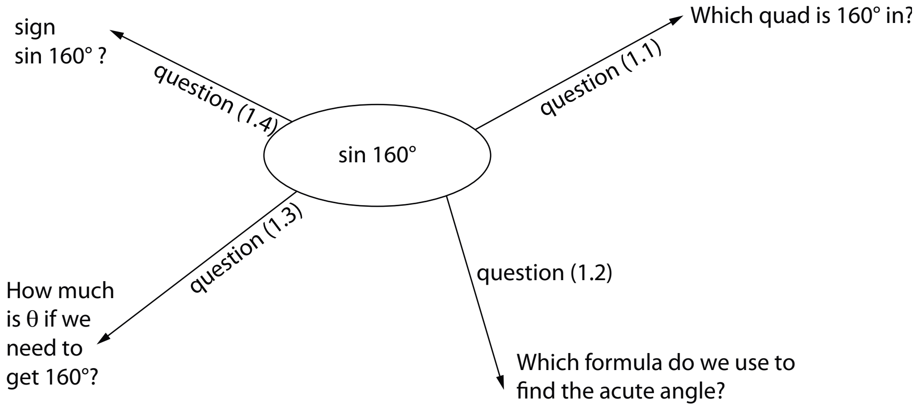
To take an angle back into one of the first 4 quadrants we simply add or subtract multiples of 360°.

$$\text{In general} \left\{ \begin{array}{l} \tan(k{\cdot}360° + \alpha) = \tan\alpha \\ \sin(k{\cdot}360° + \alpha) = \sin\alpha \\ \cos(k{\cdot}360° + \alpha) = \cos\alpha \end{array} \right\} \begin{array}{l} \text{where } k \in \mathbb{N}; \\ \alpha \in (0°; 360°) \end{array}$$

Example

Example 1

1. Reduce sin 160° to the sine of an acute angle:



This flow diagram will help you to obtain the correct solution by asking you 4 specific questions.

Solution

Solution

1.1 160° is in Quad II.

1.2 In Quad II we always use $180° - \theta$

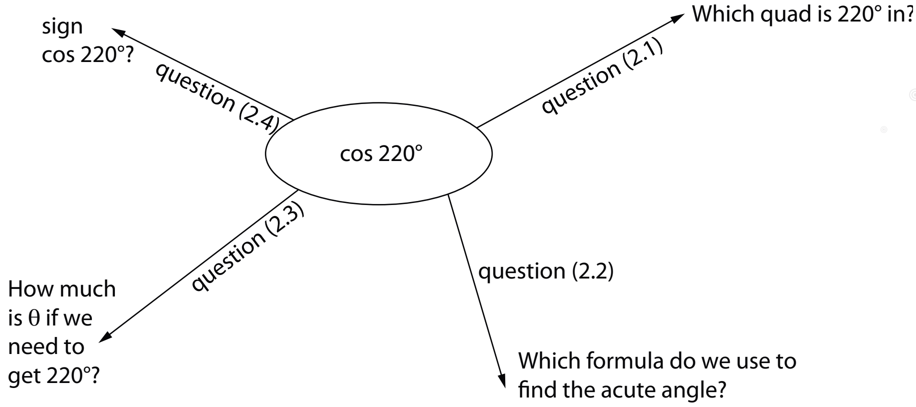
1.3 $180° - 20° = 160° \quad \therefore \theta = 20°$

1.4 $\sin 160° > 0$

Now we can write: $\sin 160° = \sin(180° - 20°) = \sin 20°$

Example 2

Reduce cos 220° to an acute angle:



Solution

2.1 220° is in Quad III

2.2 In Quad III we always use $180° + \theta$

2.3 $180° + 40° = 220° \quad \therefore \theta = 40°$

2.4 $\cos 220° < 0$

Now we can write: $\cos 220° = \cos(180° + 40°) = -\cos 40°$

Example 3

Reduce cos 1 230° to an acute angle.

1 230° is larger than 360°. Here we must first write 1 230° as $k \cdot 360° + \alpha$.

What is k? What is α?

Solution

$1\,230° = 3(360°) + 150°$. Easy way is take away 360° from the given angle until you reach an angle between 0° and 360°.

$\therefore \cos 1\,230° = \cos[3(360°) + 150°] = \cos 150° = \cos(180° - 30°) = -\cos 30°$

Activity 3

Remember to ask yourself the 4 questions which appear in the worked examples.

1. Reduce to a trig ratio of an acute angle:

1.1 sin 320°

....

1.2 cos 140°

....

1.3 tan 310°

....

1.4 sin 640°

....

2. Simplify without using a calculator.

2.1 $\frac{\tan 140°.\cos 320°}{\sin 220°}$

2.2 $\tan^2 300°$

2.3 $\frac{\tan 135°}{\sin 150° + \cos 300°} + \cos^2 240°$

3. If $\sin 16° = k$, determine the following in terms of k.

3.1 sin 344°

3.2 sin 196°

3.3 sin 164°

3.4 sin 376°

3.5 sin 736°

3.6 cos 16°

3.7 tan 16°

Example

1. Simplify to 1 trig ratio:

$\frac{\sin(180° + \theta) \cdot \sin(360° - \theta)}{\sin(180° - \theta) \cdot \tan(180° - \theta)}$

$= \frac{-\sin\theta \cdot (-\sin\theta)}{\sin\theta \cdot (-\tan\theta)}$

$= \sin\theta \div (-\tan\theta)$

$\therefore \sin\theta \div \frac{-\sin\theta}{\cos\theta}$

$= \sin\theta \times \frac{-\cos\theta}{\sin\theta} = -\cos\theta$

- In which quadrant will each of the angles be found?
  $180° + \theta$ is in Quad III
  $360° - \theta$ is in Quad IV
  $180° - \theta$ is in Quad II
- Find the sign of each trig ratio:
  $\sin(180° + \theta) = -\sin\theta$
  $\sin(360° - \theta) = -\sin\theta$
  $\sin(180° - \theta) = \sin\theta$
  $\tan(180° - \theta) = -\tan\theta$
- Substitute each reduction formula into the fraction.
- Use brackets to avoid confusion with operations.
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (Identity)

2. For which value(s) of k is $k^2 + \sin 120°.\cos 1\,110° = \frac{\cos(180° + \theta)}{\cos(180° - \theta)}$

$k^2 = \frac{\cos(180° + \theta)}{\cos(180° - \theta)} - \sin 120°.\cos 1\,110°$

$k^2 = \frac{-\cos\theta}{-\cos\theta} - [\sin(180° - 60°).\cos(3 \times 360° + 30°)]$

$k^2 = 1 - (\sin 60°.\cos 30°)$

$k^2 = 1 - \left(\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}\right)$ * Special ratios

$k^2 = 1 - \frac{3}{4}$

$k^2 = \frac{1}{4}$ $\therefore k = \pm\frac{1}{2}$

Activity 4

1. Simplify: $\frac{\sin(180° - x) \cdot \tan(360° - x)}{\tan(360° + x) \cdot \sin(360° - x)}$

....

2. Prove that: $\frac{1}{1 - \cos(180° - x)} + \frac{1}{1 - \cos(360° - x)} = \frac{2}{\sin^2 x}$

....

....

....

....

3. Simplify: $\frac{\tan(180° - \beta) \cdot \sin(180° + \beta) - \cos(180° + \beta)}{\frac{1}{\cos(360° - \beta)}}$

....

....

....

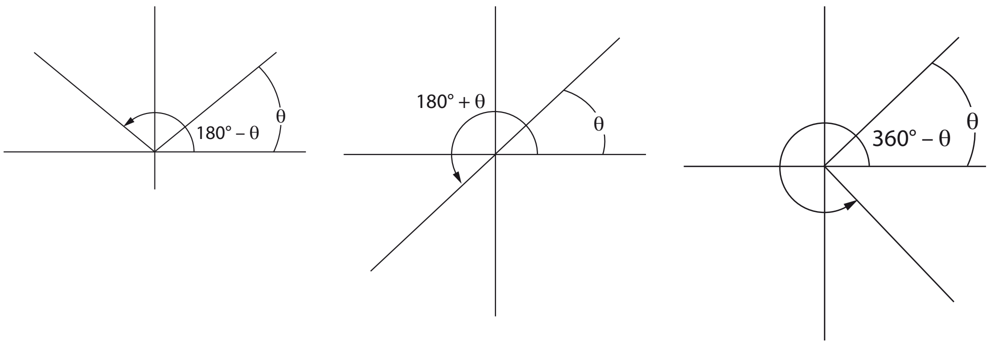
4. Prove that:

$[\cos(360° - A)][\cos(360° + A) + \sin(180° + A) \cdot \tan(180° + A)]$
$= 2\cos^2(180° + A) - 1$

RECAP: Reduction Formula

So far we have worked with 2 concepts:

1. If an angle, θ, rotated in an anti-clockwise direction has a terminal ray OP and OP = r with P(x; y) then $\sin\theta = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$; and $\tan\theta = \frac{y}{x}$
2. Angles occur in 4 quadrants, and the trig ratios of these angles can be written as a ratio of an acute angle, θ. Remember that all angles in the first quadrant are already acute angles.



$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = \cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = \tan\theta$	$\tan(360° - \theta) = -\tan\theta$

There are 3 more formulae

1. Use the calculator to complete the table below: (round off to 1 dec place).

sine 60° = 0,9	cosine 30° = 0,9
sine 30° = ……	cosine 60° = ……
sine 10° = ……	cosine 80° = ……
sine 20° = ……	cosine 70° = ……
sine 65° = ……	cosine 25° = ……

1.1 What do you notice?

1.2 Can you predict which trig ratio will equal sin θ?

Solution

1.1 We notice that the 2 trig ratios in the same row are EQUAL. The one ratio is a 'sine'-ratio and the other one a 'cosine'-ratio. In each row, the sum of the 2 angles is 90° (ie complementary).

1.2 With the above observation we can predict that

2 functions are co-functions

$$\text{sine}\ \theta = \text{cosine}\ (90° - \theta)$$

2 angles are complementary

We say this is the "Co-Co" Rule:

- $\sin \theta = \cos (90° - \theta)$
- $\cos \theta = \sin (90° - \theta)$

Example

Without a calculator, express each ratio as a ratio of 20°

1.	sin 70°	2.	cos 70°
3.	sin 110°	4.	sin 250°
5.	sin 290°	6.	sin 380°

Solution

1. $\sin 70° = \cos (90° - 20°) = \cos 20°$
2. $\cos 70° = \sin (90° - 20°) = \sin 20°$
3. $\sin 110° = \sin (180° - 70°) = \sin 70° = \cos 20°$
4. $\sin 250° = \sin (180° + 70°) = -\sin 70° = -\cos 20°$
5. $\sin 290° = \sin (360° - 70°) = -\sin 70° = -\cos 20°$
6. $\cos 430° = \cos (360° + 70°) = \cos 70° = \sin 20°$

Activity 5

1. Without a calculator simplify: $\frac{\cos 250° \cdot \tan 315°}{\sin 200°}$

2. If $\sin 80° = a$, express each of the following in terms of a

2.1 cos 10°

2.2 cos 190°

2.3 sin 10°

2.4 cos 350°

2.5 cos 530°

2.6 sin 280°

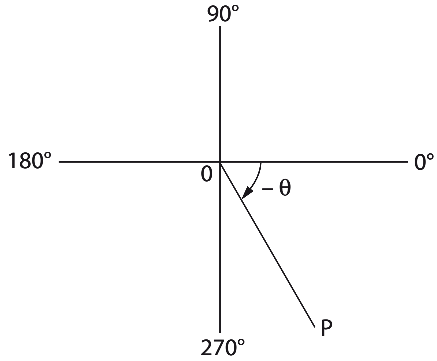
Note that: $\sin(90° + \theta) = \cos\theta$ AND $\cos(90° + \theta) = -\sin\theta$

Negative angles

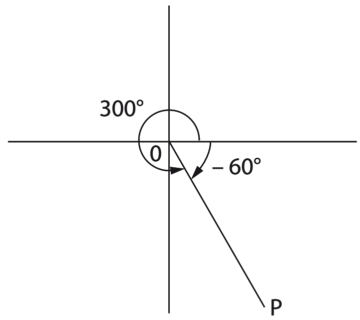
1. Definition of a negative angle:

   An angle rotated in a clockwise direction has rotated in the opposite direction to the positive angle (anti-clockwise).

   ∴ The negative sign indicates direction only. $-\theta$ has its terminal ray in Quad IV, if θ is acute



2.1 Each negative angle will have a positive angle which shares the same terminal ray OP. When 2 angles share a terminal ray, we call them coterminal angles.
∴ –60° and 300° are coterminal ∠'s, since they share OP.

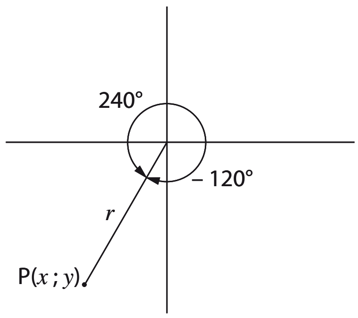


2.2 –120° and 240° are coterminal ∠'s.

$\sin(-120°) = \frac{y_P}{r}$

$\sin(240°) = \frac{y_P}{r}$

$\therefore \sin(-120°) = \sin 240°$

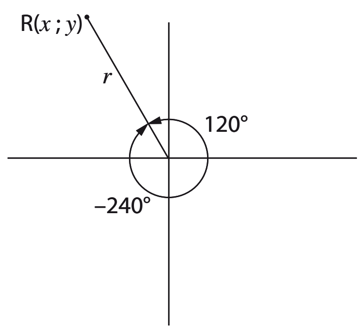


2.3 –240° and 120° are coterminal ∠'s.

$\cos 120° = \frac{x}{r}$

$\cos(-240°) = \frac{x}{r}$

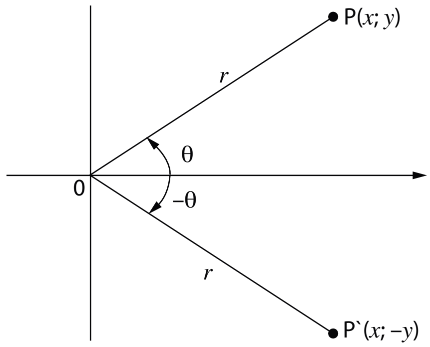
$\therefore \cos 120° = \cos(-240°)$



∴ Trig Ratios of Coterminal Angles are **EQUAL**.

2.5 Let OP rotate $\theta°$ (θ acute) where $x > 0$ and $y > 0$.

Let OP′ rotate $-\theta°$. Using the definitions of the trig ratios:



$\sin\theta = \frac{y}{r}$	$\sin(-\theta) = \frac{-y}{r}$
$\cos\theta = \frac{x}{r}$	$\cos(-\theta) = \frac{x}{r}$
$\tan\theta = \frac{y}{x}$	$\tan(-\theta) = \frac{-y}{x}$

From the above table we can draw the summary:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

Example

1. Simplify: $\sin(-\theta) + \sin(180° + \theta) + \cos(90° - \theta)$

$-\sin\theta - \sin\theta + \sin\theta$

$= -\sin\theta$

2. Simplify: $\cos(-\theta) - \cos(180° + \theta) + 3\sin(90° - \theta)$

$= \cos\theta - (-\cos\theta) + 3\cos\theta$

$= 2\cos\theta + 3\cos\theta = 5\cos\theta$

Activity 6

1. Simplify: $\dfrac{\cos(90° - x)\cdot\tan(360° - x)\cdot\sin(-x)}{\cos(90° + x)\cdot\sin(90° + x)\cdot\tan(-x)}$

2. Prove: $\dfrac{\cos(90° - x)}{\sin(90° + x)} - 3\cos(360° - x)\cdot\tan(-x) = \tan x + 3\sin x$

Trig equations

Overview

In this lesson you will:

- Use trigonometric graphs to solve sin θ = p; cos θ = p; tan θ = p.
- Define "general solutions"
- Use a calculator to solve equations
- solve trigonometric equations in a given domain

Pay attention to notation: General solution is always written with infinite periods: "$k\cdot 360°$", etc. where $k \in \mathbb{Z}$

In the last section we have seen that sin 30° = sin 150° = sin (390°) = sin (510°) = sin (–210°) = 0,5.

This shows that there are an infinite number of angles which give the same trig ratio.

In this section we will be solving a trig equation: which simply means that we would like to find ALL the possible angles, θ, for a given trig ratio.

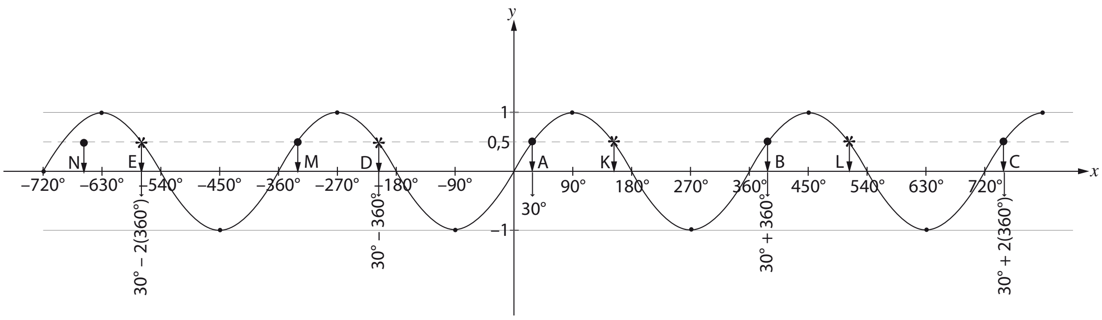
If we use the above example: let's say sin θ = 0,5, what is θ?

θ = 30° or θ = 150° or θ = 390° or θ = 510° or θ = –210° or θ = –330° or θ = … etc.

We will be finding a way of writing all the angles in a "compact version", instead of listing the infinite possibilities (this is tedious and takes forever)

Trig equations involving the "sin" -ratio

To help us develop the "compact version" of expressing all the angles for a specific trig ratio, we will start by looking at the graph of $y = \sin\theta$:



One of the features of the sine graph is its period. The period of the sine graph is 360°, which means that e sine of all angles 360° apart will have the same value.

Place your finger on the dot on the curve. Trace your way to the next dot. How many degrees have you moved?

Keep tracing the path with your finger to the various dots. How many degrees do you move from dot to dot?

If the dot at A represents 30°, then moving 360° to the right means that the angle at B = 30° + 360°, and moving a furter 360° to the right means the angle at C = 30° + 2(360°). Moving 360° to the left means that the angle at M = 30° – 360°, and moving a further 360° to the left means that the angle at N = 30° – 2(360°).

We are beginning to see a pattern emerge: If we know 1 angle, then the rest can be calculated by adding multiples of 360° (both positive and negative). In maths we write this as $k\cdot 360°$ where $k \in \mathbb{Z}$ ie. $k \in \{\ldots -3; -2; -1; 0; 1; 2; \ldots\}$

In general we say that if $\sin\theta = 0{,}5$ then $\theta = 30° + k{\cdot}360°$ where $k \in \mathbb{Z}$

BUT have we found ALL the angles? NO! Look on your graph – We have not yet found the angles at K; L; M and N.

If we know that the angle at A = 30° can we calculate the size of K? Yes. The angle at K = 180° – 30° = 150°.

$\therefore$ At K: $\theta = 150°$ which means that using the periodicity of the graph we can calculate the rest of the angles.

At L: $\theta = 150° + 360° = 510°$

At D: $\theta = 150° - 360° = -210°$

At E: $\theta = 150° - 2(360°) = -570°$

$\therefore$ In general $\theta = 150° + k{\cdot}360°$ where $k \in \mathbb{Z}$.

All the dots and stars represent all the angles where $\sin\theta = 0{,}5$.

The GENERAL SOLUTION will be the set of all the dots and stars.

ie. If $\sin\theta = 0{,}5$ then the general solution is: $\theta = 30° + k{\cdot}360°$ OR $\theta = (180° - 30°) + k{\cdot}360°$ where $k \in \mathbb{Z}$ $\theta = 150° + k{\cdot}360°$

How do we find the 1st angle? i.e. How did we find that at A: $\theta = 30°$?

USE A CALCULATOR! Enter [$\sin^{-1}$ (0,5)] on your calculator, which will display an answer of 30°.

The calculator will only give you 1 ANGLE. We have to calculate the rest of the angles using the general solution.

calc ∠ stands for the angle given by the calculator.

In general: If $\sin\theta = k$ then $\theta = \text{calc}\angle + k{\cdot}360°$ or $\theta = (180° - \text{calc}\angle) + k{\cdot}360°$, $k \in \mathbb{Z}$

Example

1. Find the general solutions to:

1.1 $\sin\theta = 0{,}17$

1.2 $\sin\theta = -0{,}5$

2. Solve for θ if $\frac{\sin\theta}{2} = -0{,}4$ and $\theta \in (-180°; 180°)$

Solution

- Write down the formulae for the General Solutions:
  $\theta = \text{calc}\angle + k{\cdot}360°$ or
  $\theta = 180° - \text{calc}\angle + k{\cdot}360°$, $k \in \mathbb{Z}$
- Use calc to find the angle [$\sin^{-1}$ (0,17)]

1.1 $\sin\theta = 0{,}17$

$\theta = 9{,}8° + k{\cdot}360°$ or $\theta = 180° - 9{,}78\ldots° + k{\cdot}360°$, $k \in \mathbb{Z}$

$= 170{,}2° + k{\cdot}360°$

$\therefore \theta = 9{,}8° + k{\cdot}360°$ or $\theta = 170{,}2° + k{\cdot}360°$, $k \in \mathbb{Z}$

1.2 $\sin\theta = -0{,}5$

$\therefore \theta = \text{calc}\angle + k{\cdot}360°$ or $\theta = (180° - \text{calc}\angle) + k{\cdot}360°$, $k \in \mathbb{Z}$

$\therefore \theta = -30° + k{\cdot}360°$ $\theta = (180° - (-30°)) + k{\cdot}360°$

$\therefore \theta = -30° + k{\cdot}360°$ or $\theta = 210° + k{\cdot}360°$

2. $\frac{\sin\theta}{2} = -0{,}4$ means that $\sin\theta = -0{,}8$

$\therefore \sin\theta = -0{,}8$

$\therefore \theta = \text{calc}\angle + k{\cdot}360°$ or $\theta = (180° - \text{calc}\angle) + k{\cdot}360°, k \in \mathbb{Z}$

$\therefore \theta = -53{,}13\ldots° + k{\cdot}360°$ $\quad\theta = (180° - (-53{,}13\ldots°)) + k{\cdot}360°$

$\theta = 233{,}1\ldots° + k{\cdot}360°$

BUT in this question we are given a domain: $\theta \in (-180°; 180°)$

$\therefore$ we only want the angles which fall between –180° and 180°.

If $\theta = -53{,}13\ldots° + k{\cdot}360°$ or $\theta = 233{,}1\ldots° + k{\cdot}360°$

then	$\theta = -53{,}1°$ if $k = 0$	$\theta = 233{,}1°$ if $k = 0$
	$\theta = 306{,}9°$ if $k = 1$	$\theta = 593{,}1°$ if $k = 1$
	$\theta = -413{,}1°$ if $k = -1$	$\theta = -126{,}9°$ if $k = -1$
	etc.	etc.

FROM THE ABOVE SOLUTIONS WE CAN SEE THAT:

ONLY $\theta = -53{,}1°$ and $\theta = -126{,}9°$ fall in the domain (–180°; 180°)

Solution set: $\theta = -53{,}1°$ or $\theta = -126{,}9°$

Activity

Activity 7

1. Find the general solution if $\sin\theta = -0{,}625$

....

....

2. Find the values of θ if $\sin\theta = 0{,}328$ and $\theta \in [-360°; 180°]$

....

....

....

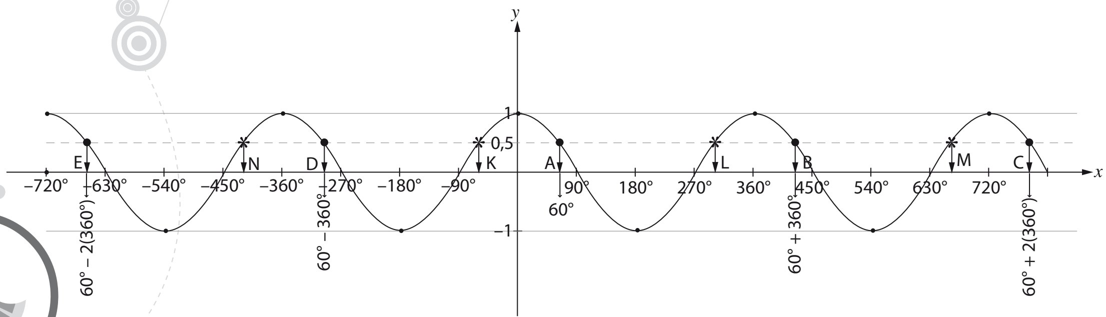
Trig equations involving the "cos"-ratio

We noticed earlier that:

cos (60°) = cos (300°) = cos (420°) = cos (660°) = cos (–60°) = cos (–300°) = 0,5

There are an infinite number of angles which give the same trig ratio. We would like to develop the general solution for $\cos\theta = 0{,}5$. How do we find ALL the angles which give $\cos\theta = 0{,}5$?

Let us begin by sketching the graph of $y = \cos\theta$.



From the graph we see that,

The GENERAL SOLUTION for the "dots" and "stars" is:
If $\cos\theta = 0{,}5$ then $\theta = 60° + k \cdot 360°$ OR $\theta = -60° + k \cdot 360°$ where $k \in \mathbb{Z}$

Lets recap. How did we find the 1st angle at A? How do we know that at A, $\theta = 60°$?

USE A CALCULATOR! Enter [$\cos^{-1}(0{,}5)$] on your calculator, which will display an answer of 60°.

The calculator will only give you [1 ANGLE] for the 1st part of the general solution. For the 2nd part of the general solution we use the additive inverse.

[calc ∠] → stands for the angle given by the calculator.

In general:	If $\cos\theta = k$ then $\theta = \text{calc}\angle + k \cdot 360°$ or $\theta = -(\text{calc}\angle) + k \cdot 360°$, $k \in \mathbb{Z}$

Example

Example

1. Find the general solution to

1.1 $\cos\theta = 0{,}17$

1.2 $\cos\theta = -0{,}5$

2. Solve for θ if $2\cos\theta = 1{,}8$ and $\theta \in (-360°; 0°)$

3. Find the general solution if $\cos(2x + 30°) = 0{,}17$

1.1 $\cos\theta = 0{,}17$

$\theta = 80{,}2° + k \cdot 360°$

or

$\theta = -80{,}2° + k \cdot 360°$, $k \in \mathbb{Z}$

- Write down the formula for the general solution. $\theta = \text{calc}\angle + k \cdot 360°$ or $\theta = -(\text{cal}\angle) + k \cdot 360°$ where $k \in \mathbb{Z}$
- Use the calculator to find (cal ∠) i.e. $\cos^{-1}(0{,}17)$

1.2 $\cos\theta = -0{,}5$

$\therefore \theta = 120° + k \cdot 360°$ or $\theta = -120° + k \cdot 360°$, $k \in \mathbb{Z}$

2. $2\cos\theta = 1{,}8$

$\therefore \cos\theta = 0{,}9$

$\therefore \theta = 25{,}8° + k \cdot 360°$ or $\theta = -25{,}8° + k \cdot 360°$, $k \in \mathbb{Z}$

BUT we are asked to find θ in the given domain $(-360°; 0°)$, so now we will have to substitute values of k which will give angles in the interval $(-360°; 0°)$.

If $k = 0$	$\theta = 25{,}8°$	or $\theta = -25{,}8°$
If $k = 1$	$\theta = 385{,}8°$	or $\theta = 334{,}2°$
If $k = -1$	$\theta = -334{,}2°$	or $\theta = -385{,}8°$

We can easily see that only 2 angles satisfy the given interval:

Solution set: $\theta = -25{,}8°$ or $\theta = -334{,}2°$

3. $\cos(2x + 30°) = 0{,}17$

$\therefore 2x + 30° = 80{,}2° + k \cdot 360°$ or $2x + 30° = -80{,}2° + k \cdot 360°$, $k \in \mathbb{Z}$

$2x = 50{,}2° + k \cdot 360°$ $2x = -110{,}2° + k \cdot 360°$

$x = 25{,}1° + k \cdot 180°$ $x = -55{,}1° + k \cdot 180°$

Activity 8

1.. Find the general solution if $\cos\theta = -0{,}625$

2.. Solve for θ if $\cos\theta = 0{,}82$ and $\theta \in (-90°; 360°)$

Trig equations involving the "tan" -ratio

We noted earlier that

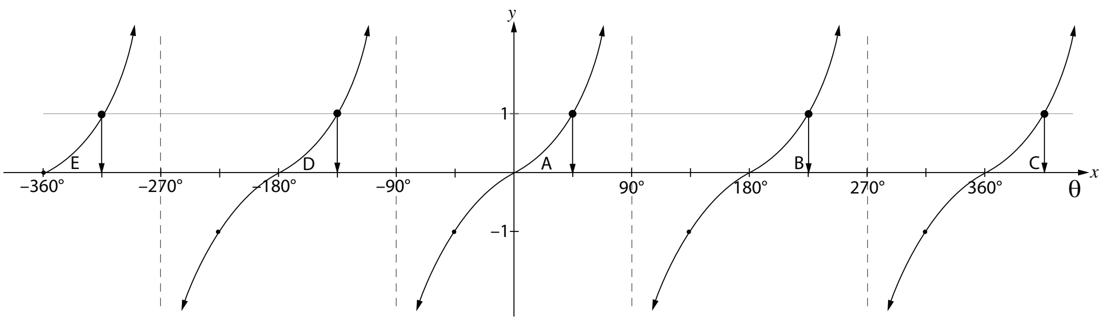
tan 45° = tan 225° = tan 405° = tan 585° = tan(–315°) = tan(–135°) = 1

There are an infinite number of angles which give the same trig ratio.

We would now like to find the general solution for $\tan\theta = 1$.

How do we find all the angles which give $\tan\theta = 1$?

Let us begin by sketching the graph of $y = \tan\theta$.



The above graph represents $y = \tan\theta$.

This graph is completely different from the sine and cos graphs.

The graph is DISCONTINUOUS. It has ASYMPTOTES every 180°.

The vertical lines at $\theta = 90°$; $\theta = -90°$; $\theta = 270°$ and $\theta = -270°$ are asymptotes. The tan curve approaches the asymptotes but never crosses the asymptotes.

Since the asymptotes repeat themselves every 180°, we can see that the tan curve has a PERIOD OF 180°.

Since $\tan\theta = 1$, then at A, $\theta = 45°$

If we move 180° to the right then at B, $\theta = 45° + 180°$, and

if we move a further 180° to the right, then at C, $\theta = 45° + 2(180°)$.

If we move 180° to the left then at D, $\theta = 45° - 180°$, and

if we move a further 180° to the left, then at E, $\theta = 45° - 2(180°)$.

The pattern which emerges is that if we want to find the angles represented by the 'dots' we add multiples of 180° to 45°.

In general we say that if $\tan\theta = 1$, then $= 45° + k \cdot 180°$ where $k \in \mathbb{Z}$

The above formula finds all the angles, so therefore the above formula is the solution.

Lets recap. How do we find the 1st angle at A? How do we know that at A, $\theta = 45°$?

Use a calculator! Enter [$\tan^{-1}(1)$] on your calculator, which will display an answer of 45°.

calc $\angle$ $\rightarrow$ stands for the calculator angle.

In general if $\tan\theta = p$

then $\theta = \text{calc}\angle + k{\cdot}180°, k \in \mathbb{Z}$

Example

1. Find the general solution to:

1.1 $\tan\theta = 5{,}17$

1.2 $\tan\theta = -1$

2. Solve for θ if $\tan\theta + 2 = 9{,}5$ and $\theta \in (-180°; 180°)$

Solution

- Write down the formula for the general solution $\theta = \text{calc}\angle + k{\cdot}180°$
- Use the calculator to find the angle [tan-1 (5,17)]

1.1 $\tan\theta = 5{,}17$

$\theta = 79{,}1° + k{\cdot}180°, k \in \mathbb{Z}$

1.2 $\tan\theta = -1$

$\theta = \text{calc}\angle + k{\cdot}180°$ $\theta = -45° + k{\cdot}180°, k \in \mathbb{Z}$

2. $\tan\theta + 2 = 9{,}5$

$\tan\theta = 7{,}5$

$\theta = 82{,}4° + k{\cdot}180°, k \in \mathbb{Z}$

if $k = 0$ then $\theta = 82{,}4°$

if $k = -1$ then $\theta = -97{,}6°$

Set: $\theta = 82{,}4$ or $\theta = -97{,}6°$

Activity 9

1. Find the general solution for each equation:

1.1 $\sin\theta = -0{,}21$

....

1.2 $\cos\frac{\theta}{2} = 3{,}21$

....

1.3 $\cos\frac{\theta}{2} = -0{,}21$

....

1.4 $\tan(\theta + 30°) = 20{,}21$

....

1.5 $4 + \tan\frac{\theta}{3} = -2{,}12$

2. If $\theta \in (-360°; 180°)$, find θ if $2\cos(\theta - 20) = 0{,}632$

3. $-5 \tan \theta = -5{,}05$ find θ if $\theta \in (-90°; 360°)$

Sine; cosine and area rules

Overview

In this lesson you will:

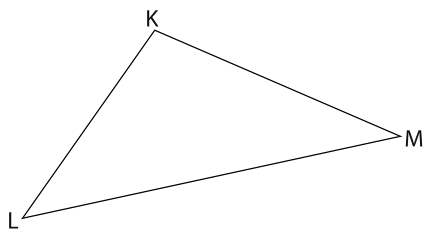
- Learn how to calculate the sides and angles in a non right angled Δ
- Learn about the "ambiguous" case of the sine rule
- Use a calculator to solve triangles
- Use angles of depression and elevation to solve problems in 2-dimensions.

How to find the sides and angles of a non-right angled △.

SINE RULE

In any triangle, the sine rule is a set of 3 equivalent fractions (with side opposite the angle).

i.e.



$$\frac{KM}{\sin L} = \frac{LM}{\sin K} = \frac{KL}{\sin M}$$

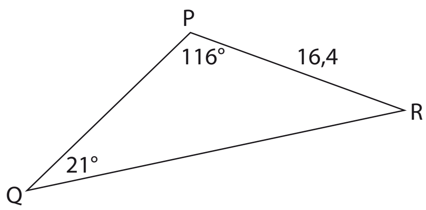
$$\text{or } \frac{\sin L}{KM} = \frac{\sin K}{LM} = \frac{\sin M}{KL}$$

The sine rule is very useful to calculate an angle or a side of a triangle when the given information has a side and an angle opposite each other.

Example

Example

Solve △PQR where $\hat{P} = 116°$; $\hat{Q} = 21°$ and PR = 16,4



LIBERTY LIFE

$$\frac{QR}{\sin P} = \frac{PR}{\sin Q}$$

$$\frac{QR}{\sin 116°} = \frac{16{,}4}{\sin 21°}$$

$$QR = \frac{(16{,}4)\sin 116°}{\sin 21°}$$

QR = 41,1 (rounded off to 1 decimal place)

- Always draw a diagram
- Look at the position of the given information: side PR is opposite $\hat{Q}$. $\hat{P}$ is opposite QR.
- Since we don't know how long QR is we can use the sine rule
- Substitute the given values
- Use a calculator to find QR

Solve △PQR, means that we must find all the unknown angles and sides of the given △.

$\hat{R} = 180° - (116° + 21°) = 43°$ (∠ sum in △ is 180°)

Side PQ is opposite $\hat{R}$, so we can use the sine rule to find PQ:

$$\frac{PQ}{\sin R} = \frac{PR}{\sin Q}$$

$$\therefore \frac{PQ}{\sin 43°} = \frac{16{,}4}{\sin 21°}$$

$$\therefore PQ = \frac{16{,}4(\sin 43°)}{\sin 21°}$$

$\therefore PQ = 31{,}2$

Activity 10

Activity

Draw a diagram for each question:

1. In △ABC, $\hat{B} = 63°$; $\hat{C} = 37°$ and AC = 35. Solve △ABC.

....

....

....

....

....

....

....

....

2. In △PQR, $\hat{Q} = 75°$; $\hat{R} = 42°$ and QR = 35. Solve △PQR.

....

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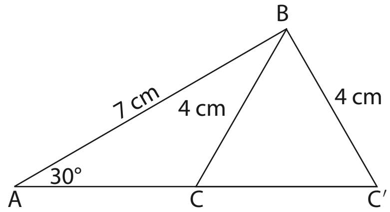
....

Special case

Accurately draw $\triangle ABC$ with $\hat{A} = 30°$; AB = 7 cm and BC = 4 cm

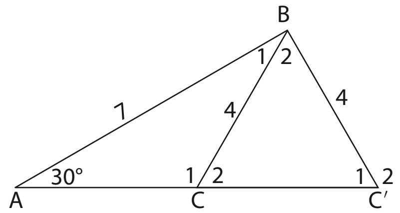


$\triangle ABC$ and $\triangle ABC'$ both have the given dimensions.

This situation arises because the side opposite the angle is the shorter of the two given sides. In other words, if $\triangle ABC$ had $\hat{A} = 30°$; AB = 4 cm and BC = 7 cm, then we would only get 1 triangle, because BC is opposite the given angle but it is longer than AB.

We say that when 2 triangles exist from the same given information that the situation is ambiguous.

Lets solve both triangles.



In $\triangle ABC'$; $\frac{\sin C_1'}{AB} = \frac{\sin A}{4}$ (Sine Rule)

$\therefore \sin C_1' = \frac{(\sin 30°)(7)}{4}$

$= 0{,}875$

$\hat{C}_1' = 61{,}0°$

Since $\triangle BCC'$ is isoceles: $BC = BC' = 4$

$\hat{C}_2 = \hat{C}_1' = 61°$

$\therefore \hat{C}_1 = 119°$ (adj. $\angle$° on str. line AC)

In $\triangle ABC'$: $\hat{B}_1 + \hat{B}_2 = 89°$ ($\angle$ sum in $\triangle$) $\therefore \frac{AC^1}{\sin(\hat{B}_1 + \hat{B}_2)} = \frac{BC^1}{\sin 30°}$

$AC^1 = \frac{4}{\sin 30°} \cdot \sin 89°$

$AC' = 8$ units

In $\triangle ABC$: $\hat{B}_1 = 31°$ ($\angle$ sum in $\triangle$) $\therefore \frac{AC}{\sin B_1} = \frac{BC}{\sin 30°}$

$AC = \frac{4}{\sin 30°} \cdot \sin 31°$

$AC = 4{,}1$ units

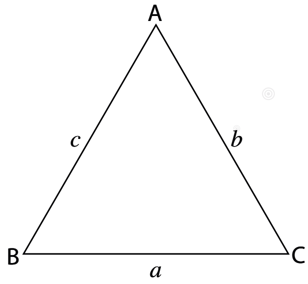
COSINE RULE

The Cosine Rule:

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

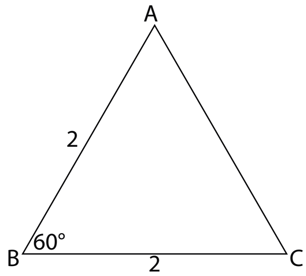


This rule is useful when:

1. We are given SAS: 2 sides adjacent to an angle and we want to find the 3rd side.
2. We are given SSS: 3 sides of a △ and we want to find an angle.

For example:

1.



If AB = BC = 2 and $\hat{B} = 60°$ then

$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos B$

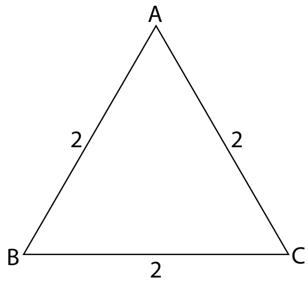
$AC^2 = (2)^2 + (2)^2 - 2(2)(2) \cdot \cos 60°$

$\therefore AC^2 = 4$

$\therefore AC = 2$

> We always solve for the side opposite the given angle when we have SAS.

2.



If AB = BC = AC = 2 and we want to calculate the size of $\hat{A}$, then $BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos A$

$(2)^2 = (2)^2 + (2)^2 - 2(2)(2) \cdot \cos A$

$\frac{4 - 4 - 4}{-8} = \cos A$

$+\frac{1}{2} = \cos A$

$\therefore A = 60°$

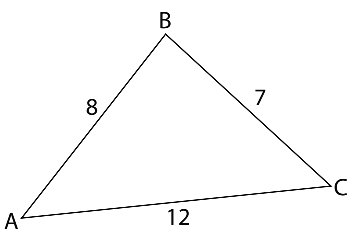
> When we are given 3 sides we can solve for any of the 3 angles. The angle that we find is always opposite the side that appears first in the formula.

Example

Example

1. Find the largest angle in △ABC where AB = 8, BC = 7, and AC = 12

   The largest angle will be opposite the largest side, and the smallest angle will be opposite the smallest side.



$\hat{B}$ is opposite side AC, therefore $\hat{B}$ is the largest angle.

Since we have been given 3 sides, we can use the cosine rule to find $\hat{B}$.

$\therefore AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos B$

$(12)^2 = (8)^2 + (7)^2 - 2(8)(7)\cos B$

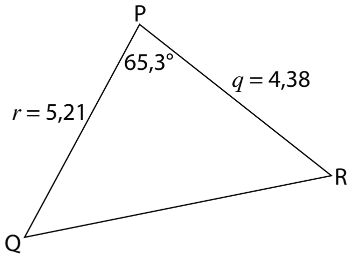
$\frac{12^2 - 8^2 - 7^2}{-2(8)(7)} = \cos B$

$\frac{31}{-112} = \cos B$

Since cos B is a negative ratio, this means that B is obtuse (in the 2nd quad)

$\therefore B = 106{,}1°$

2.



Find the 3rd side of $\triangle PQR$ if $q = 4{,}38$, $r = 5{,}21$ and $P = 65{,}3°$

Since we have been given SIDE; ANGLE; SIDE (SAS), we can use the Cosine Rule as follows:

$QR^2 = PQ^2 + PR^2 - 2PQ \cdot PR \cos P$

$QR^2 = (5{,}21)^2 + (4{,}38)^2 - 2(5{,}21)(4{,}38)\cos 65{,}3°$

$\therefore QR^2 = 27{,}25\ldots$

$\therefore QR = 5{,}2$ (correct to 1 decimal place)

Activity

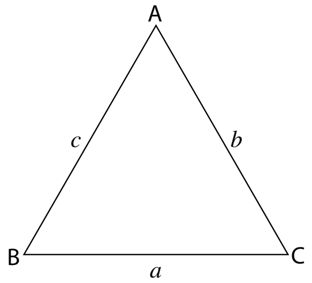
Activity 11

Solve the following triangles

1. $\triangle ABC$ where AB = 80; BC = 90 and AC = 100

2. $\triangle KLM$ where $\hat{K} = 132°$, KL = 5 and KM = 7

THE AREA RULE



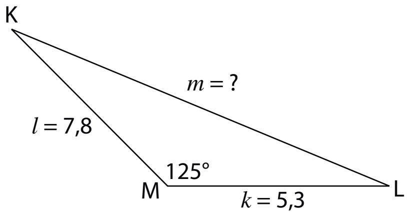
In any triangle ABC:

$$\text{area} = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}bc\sin A$$
$$= \frac{1}{2}ac\sin B$$

In order to find the area using this rule, we must be given SAS.

Example

Determine the area of ΔKLM in which $k = 5{,}3$, $\ell = 7{,}8$ and $\hat{M} = 125°$



Draw a diagram

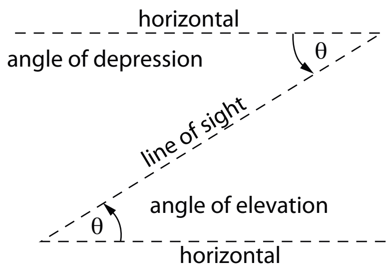
Since we have SAS, Area △KLM $= \frac{1}{2}$ ML·MK·sin M

$= \frac{1}{2}(5{,}3)(7{,}8)\sin 125°$

$= 16{,}9$ sq units (correct to 1 decimal place)

Solving problems in 2 dimensions

Angles of elevation and depression



The angle of elevation is the angle between the horizontal and the line of sight when looking up.

The angle of depression is the angle between the horizontal and the line of sight when looking down.

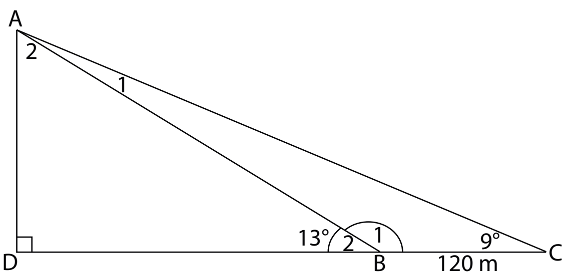
Example

A boat is sailing directly towards the foot of a cliff.

The angle of elevation of a point on the top of the cliff, and directly ahead of the boat, increases from 9° to 13° as the boat sails 120 m towards the cliff.

Find:

1. the original distance from the boat to the point on the top of the cliff.
2. the height of the cliff.



A is the top of the cliff.

C is the boat at original position.

B is the boat's new position.

We need to find AC.

1. $A_1 = 4°$ (ext $\angle$ of $\triangle$: $\hat{C} + \hat{A}_1 = \hat{B}_2$) ; $\hat{B}_1 = 167°$ (str. $\angle$)

Once you have calculated these angles, 'draw' them into your diagram.

Using the Sine Rule in $\triangle ABC$: $\frac{AC}{\sin B_1} = \frac{BC}{\sin A_1}$

$\therefore \frac{AC}{\sin 167°} = \frac{120}{\sin 4°}$

AC = 357,7 metres (correct to 1 decimal place)

2. In $\triangle ADC$

$\frac{AD}{AC} = \sin 9°$

$\therefore AD = 386{,}9766499 \cdot \sin 9°$

$= 60{,}5$m (correct to 1 decimal place)

Example

Example

2. The side of a triangular field AB, of length 100 m, lies in a direction north-west of A.

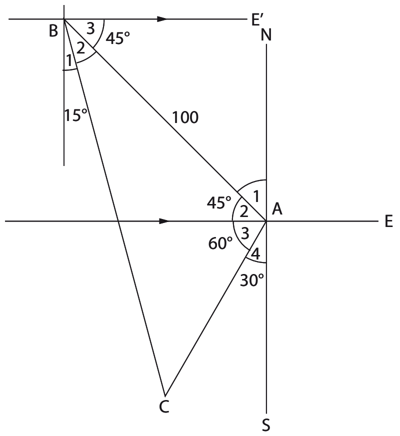
From A, the corner C is in a direction S30°W and from B, C is a direction S15°E.

Find:

2.1 Distance from A to C

2.2 Distance from B to C

2.3 Area of the field



- NW means $\hat{A}_2 = 45°$
- S30°W means move 30° West of South
- NS$\perp$WE
- S15°E means move 15° East of South
  $\therefore \hat{B}_1 = 15°$
- BE'$\|$WE
  $\therefore \hat{B}_3 = 45°$

Solution

Solution

2.1 $\hat{B}_2 = 30°$ ($\hat{B}_1 + \hat{B}_2 + \hat{B}_3 = 90°$)

$\hat{A}_2 + \hat{A}_3 = 105°$

$\hat{C} = 45°$ ($\angle$ sum in $\triangle$)

In $\triangle ABC$, we use the Sine Rule: $\frac{AC}{\sin B_2} = \frac{AB}{\sin C}$

$\therefore \frac{AC}{\sin 30°} = \frac{100}{\sin 45°}$

AC = 70,7 metres (correct to 1 decimal place)

2.2 Using Cosine Rule:

Note: It is possible to find the length of BC using the sine rule.

$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \sin(A_2 + A_3)$

$BC^2 = 100^2 + (70{,}7)^2 - 2(100)(70{,}7)\sin(105°)$

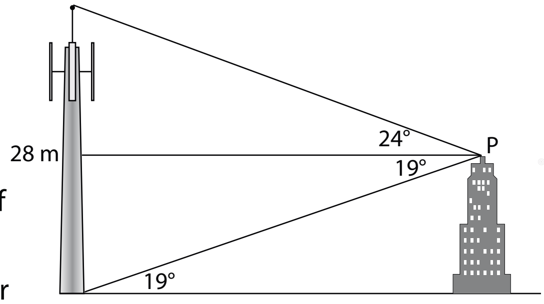
BC = 36,6 metres (correct to 1 decimal place)

2.3 Area $= \frac{1}{2}(100)(70{,}7)\sin 105° = 3414{,}5$ m^2 (correct to 1 decimal place)

Activity 12

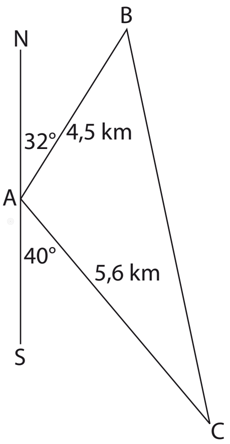
1. From a point P, on top of a building, the angle of elevation to the top of a Vodacom Tower is 24° and the angle of depression to the foot of the tower is 19°.

   If the height of the tower is 28 m, how far is the building from the tower, if they lie in the same horizontal plane.



2. A man stands on one bank of a river and finds that the angle of elevation of the top of a tree on the other bank is 18,3°. If he moves 45 m backwards in line with his first position and the tree, he finds that the angle of elevation is now 13,7°. Calculate the height of the tree and the width of the river (to the nearest metre).

3.



From an observation point A, two points B and C are in the directions N32°E and S40°E respectively. They are 4,5 km and 5,6 km from A respectively.

Calculate

a) the distance BC

b) $\hat{C}$

c) Area $\triangle ABC$

Activity

Activity 13

1. PERFORMING ROUTINE PROCEDURES

1.1 If $\sin 19° = t$, determine in terms of t

1.1.1 $\sin 161°$

1.1.2 $\cos 71°$

1.1.3 $\sin 341°$

1.1.4 $\cos 19°$

1.1.5 $\tan 71°$

1.2 Simplify without a calculator

$$\frac{\cos 135° \cdot \tan 60° \cdot \sin 315°}{\cos 300°}$$

1.3 Simplify: $\dfrac{\sin 80° \cdot \sin(360 - x) \cdot \sin(-x)}{\cos(90° - x) \cdot \cos(360° + x) \cdot \tan(180° - x) \cdot \cos 10°}$

1.4 Prove that $\dfrac{\sin\beta}{1 + \cos\beta} + \dfrac{\cos\beta}{\sin\beta} = \dfrac{1}{\sin\beta}$

1.5 Give the general solution for A if:

1.5.1 $\tan A = -0{,}758$

1.5.2 $\cos A + 0{,}32 = 1$

1.5.3 $\sin 2A = 0{,}72$

2. PERFORMING COMPLEX PROCEDURES

2.1 Simplify without a calculator:

2.1.1 $\frac{3}{2}\tan^2(-30°) - \frac{3}{2}\cos 300° - 2\sin^2(-1035°)$

2.1.2 $\sin 303° \cdot \cos 213° - \frac{1}{\tan 123°} \cdot \cos 33° \cdot \cos(-57°)$

2.2 Prove the following Identities:

2.2.1 $(\tan A + 1)\left(\frac{1}{\tan A} + 1\right) = \frac{1}{\sin A \cos A} + 2$

2.2.2 $\frac{\cos(-A)}{1 + \sin A} + \frac{1 - \sin(180° + A)}{\cos A} = \frac{2}{\cos A}$

2.2.3 $\frac{1 + \sin(-\theta)}{1 - \sin(180° + \theta)} = \left(\frac{1}{\cos\theta} - \tan\theta\right)^2$

2.2.4 $\left[\frac{1}{\sin(-\theta)} + \frac{1}{\cos(180° + \theta)}\right] \div \left[\tan\theta - \frac{1}{\tan(-\theta - 180°)}\right] = -\cos\theta - \sin\theta$

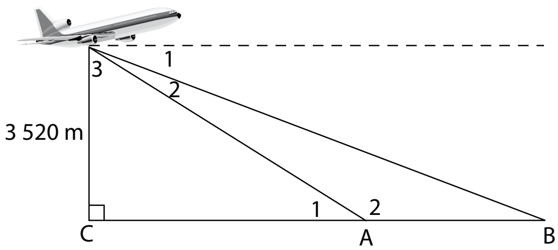
2.3 Solve for β if

$$\frac{5\cos(2\beta - 30°)}{2} = -0{,}945$$

3. PROBLEM SOLVING

3.1 From the top of a tower 95 m high, the angle of elevation of a hill is 24,3°. From the foot of the tower the angle of elevation is 32,8°. Calculate, to the nearest metre, the height of the hill.

3.2 At a particular moment the angles of depression from an aeroplane to two towns, (A & B) directly north of the aeroplane are 47,6° and 23,4° respectively. Calculate the distance between the 2 towns (in kilometres) if the aeroplane is flying at a height of 3 520 metres.



Solutions

Activity 1

1.1 $\frac{\cos 30°}{\sin 30°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

1.2 $\sin^2 60° + \cos^2 60° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

1.3 $(\cos 60° + \sin 60°)^2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{2\sqrt{3}}{4} + \frac{3}{4} = 1 + \frac{2\sqrt{3}}{4}$

$= \frac{4 + 2\sqrt{3}}{4}$

2.1 $\cos^2 60° + \sin^2 60° = 1$

$\sin^2 45° + \cos^2 45° = 1$

$\therefore$ Both expressions equal 1.

$\therefore \cos^2 60° + \sin^2 60° = \sin^2 45° + \cos^2 45°$

2.2 $(\cos 60° + \sin 60°)^2 = 1 + \frac{2\sqrt{3}}{4}$

$(\cos 45° + \sin 45°)^2 = 2$

$\therefore$ The expressions equal different values.

$\therefore (\cos 60° + \sin 60°)^2 \neq (\cos 45° + \sin 45°)^2$

2.3 The sum of the squares of each ratio is **not** the same as the sum of the ratios squared.

Activity 2

1. $\frac{\tan\theta}{1+\tan^2\theta} = \frac{\sin\theta}{\cos\theta} \div \left[1 + \frac{\sin^2\theta}{\cos^2\theta}\right]$ Write the division in a horizontal format and use BRACKETS around the divisor.

$= \frac{\sin\theta}{\cos\theta} \div \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$ Write the divisor as 1 FRACTION.

$= \frac{\sin\theta}{\cos\theta} \times \frac{\cos^2\theta}{1} = \sin\theta \cdot \cos\theta$

2. LHS: $\frac{\frac{\sin\theta}{\cos\theta}\cdot\sqrt{\cos^2\theta}}{\sin\theta}$ RHS: $\frac{1}{2} + \frac{1}{2}$
$= 1$

$= \frac{\sin\theta}{\cos\theta}\cdot\cos\theta \times \frac{1}{\sin\theta}$

$= 1$

Both expressions equal **one**.

$\therefore \frac{\tan\theta\cdot\sqrt{1-\sin^2\theta}}{\sin\theta} = \sin 30° + \cos 60°$

3. $k = \cos A + \sin A$ $\qquad p = \sin A - \cos A$

$\therefore k^2 = (\cos A + \sin A)^2$ $\qquad p^2 = (\sin A - \cos A)^2$

$\therefore k^2 = \cos^2 A + 2\sin A\cos A + \sin^2 A$ $\qquad p^2 = \sin^2 A - 2\sin A\cos A + \cos^2 A$

$\therefore k^2 = \cos^2 A + \sin^2 A + 2\sin A\cos A$ $\qquad p^2 = \sin^2 A + \cos^2 A - 2\sin A\cos A$

$\therefore k^2 + p^2 = 1 + 2\sin A\cos A + 1 - 2\sin A\cos A$

$= 2$

4.1 LHS: $\frac{\cos\beta}{\sin\beta} + \frac{\sin\beta}{\cos\beta}$ RHS: $(\sin\beta\cdot\cos\beta)^{-1}$

$= \frac{\cos^2\beta + \sin^2\beta}{\sin\beta\cos\beta}$ $\qquad = \frac{1}{\sin\beta\cos\beta}$

$= \frac{1}{\sin\beta\cos\beta}$

$\therefore$ LHS = RHS

$\therefore \frac{\cos\beta}{\sin\beta} + \tan\beta = (\sin\beta\cdot\cos\beta)^{-1}$

4.2 LHS: $\frac{3\cos^2 x + 3\sin^2 x - 1}{2 - 2\sin^2 x}$ RHS: $\frac{1}{\cos^2 x}$

$= \frac{3(\cos^2 x + \sin^2 x) - 1}{2(1 - \sin^2 x)}$

$= \frac{2}{2(\cos^2 x)} = \frac{1}{\cos^2 x} =$ RHS

4.3 LHS: $\frac{1 - \cos\beta + 1 + \cos\beta}{(1+\cos\beta)(1-\cos\beta)}$ RHS: $\frac{2}{\sin^2\beta}$

$= \frac{2}{1 - \cos^2\beta}$

$= \frac{2}{\sin^2\beta}$

$\therefore$ LHS = RHS $\qquad \therefore \frac{1}{1+\cos\beta} + \frac{1}{1-\cos\beta} = \frac{2}{\sin^2\beta}$

Activity 3

1.1 $\sin 320° = \sin(360° - 40°) = -\sin 40°$

1.2 $\cos 140° = \cos(180° - 40°) = -\cos 40°$

1.3 $\tan 310° = \tan(360° - 50°) = -\tan 50°$

1.4 $\sin 640° = \sin(360° + 280°) = \sin 280° = \sin(360° - 80°) = -\sin 80°$

2.1 $\frac{\tan 140° \cdot \cos 320°}{\sin 220°} = \frac{-\tan 40° \cdot (\cos 40°)}{-\sin 40°}$

$= \frac{\sin 40°}{\cos 40°} \cdot \cos 40° \cdot \frac{1}{\sin 40°}$

$= 1$

2.2 $\tan^2 300° = [\tan(360° - 60°)]^2 = [-\tan 60°]^2$

$= (-\sqrt{3})^2 = 3$

Note: $(-\tan 60°)^2$ is **not** the same as $-\tan^2 60°$

2.3 $\frac{\tan 135°}{\sin 150° + \cos 300°} + \cos^2 240° = \frac{-\tan 45°}{\sin 30° + \cos 60°} + \cos^2 60°$

$= \frac{-1}{\frac{1}{2} + \frac{1}{2}} + \left(\frac{1}{2}\right)^2$

$= -1 + \frac{1}{4} = -\frac{3}{4}$

3 $k = \sin 16°$

3.1 $\sin 344° = -\sin 16° = -k$

3.2 $\sin 196° = -\sin 16° = -k$

3.3 $\sin 164° = \sin 16° = k$

3.4 $\sin 376° = \sin 16° = k$

3.5 $\sin 736° = \sin 16° = k$

3.6 $\cos 16° = \sqrt{1 - \sin^2 16°} = \sqrt{1 - k^2}$

3.7 $\tan 16° = \frac{\sin 16°}{\cos 16°} = \frac{k}{\sqrt{1 - k^2}}$

Activity 4

1. $\frac{\sin x \cdot (-\tan x)}{\tan x \cdot (-\sin x)} = 1$

2. LHS: $\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x}$ RHS: $\frac{2}{\sin^2 x}$

$= \frac{1 - \cos x + 1 + \cos x}{(1 + \cos x)(1 - \cos x)}$

$= \frac{2}{1 - \cos^2 x} = \frac{2}{\sin^2 x}$

$\therefore$ LHS = RHS

3. $\frac{(-\tan\beta)(-\sin\beta) - (-\cos\beta)}{\frac{1}{\cos\beta}} = \left(\frac{\sin^2\beta}{\cos\beta} + \cos\beta\right) \div \frac{1}{\cos\beta}$

$= \frac{\sin^2\beta + \cos^2\beta}{\cos\beta} \times \cos\beta$

$= \frac{1}{\cos\beta} \times \cos\beta = 1$

4. LHS: $[\cos A][\cos A + (-\sin A)(\tan A)]$ RHS: $2\cos^2 A - 1$

$= \cos A \left[\cos A - \sin A \cdot \frac{(\sin A)}{\cos A}\right]$

$= \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$= 2\cos^2 A - 1$

Activity 5

1. $\frac{\cos 250° \cdot \tan 315°}{\sin 200°} = \frac{(-\cos 70°) \cdot (-\tan 45°)}{(-\sin 20)}$

$= \frac{(-\sin 20°)(-1)}{(-\sin 20°)} = -1$

2. $\sin 80° = a$

2.1 $\cos 10° = \sin 80° = a$

2.2 $\cos 190° = -\cos 10° = -\sin 80° = -a$

2.3 $\sin 10° = \sqrt{1 - \cos^2 10°} = \sqrt{1 - a^2}$

2.4 $\cos 350° = \cos 10° = \sin 80° = a$

2.5 $\cos 530° = \cos 170° = -\cos 10°$

$= -\sin 80°$

$= -a$

2.6 $\sin 280° = -\sin 80° = -a$

Activity 6

1. $\frac{\sin x \cdot (-\tan x)(-\sin x)}{-\sin x \cdot (\cos x)(-\tan x)} = \frac{\sin x}{\cos x} = \tan x$

2. LHS: $\frac{\sin x}{\cos x} - 3\cos x \cdot (-\tan x)$ RHS: $\tan x + 3\sin x$

$= \tan x - 3\cos x \cdot \left(\frac{-\sin x}{\cos x}\right)$

$= \tan x + 3\sin x$

$\therefore$ LHS = RHS

Activity 7

1. $\sin \theta = -0{,}625$

$\therefore \theta = -38{,}68° + k \cdot 360°$ or $\theta = 218{,}7° + k \cdot 360°, k \in \mathbb{Z}$

2. $\sin \theta = 0{,}328$

$\theta = 19{,}2° + k \cdot 360°$ or $\theta = 160{,}9° + k \cdot 360°, k \in \mathbb{Z}$

$\therefore$ Solution set: $\theta \in \{-340{,}8°; -199{,}1°; 19{,}2°; 160{,}9°\}$

Activity 8

1. $\cos \theta = -0{,}625$

$\therefore \theta = \pm 128{,}7° + k \cdot 360°; k \in \mathbb{Z}$

2. $\cos \theta = 0{,}82$

$\therefore \theta = 34{,}9° + k \cdot 360°$ or $\theta = -34{,}9° + k \cdot 360°; k \in \mathbb{Z}$

$\therefore$ Solution set: $\theta \in \{-38{,}8°; 38{,}8°; + 321{,}2°\}$

Activity 9

1.1 $\sin \theta = -0,21$

$\theta = -12,1° + k{\cdot}360°$ or $\theta = 192,1° + k{\cdot}360°; k \in \mathbb{Z}$

1.2 $\cos \frac{\theta}{2} = 3,21$

No solution as $-1 \leq \cos \theta \leq 1$ $\therefore$ Max value of $\cos \theta$ is 1.

1.3 $\cos \frac{\theta}{2} = -0,21$

$\frac{\theta}{2} = 102,1° + k{\cdot}360°$ or $\frac{\theta}{2} = -102,1° + k{\cdot}360°, k \in \mathbb{Z}$

$\therefore\ \theta = 204,2° + k{\cdot}720°$ or $\theta = -204,2° + k{\cdot}720°$

1.4 $\tan(\theta + 30°) = 20,21$

$\therefore\ (\theta + 30°) = 87,2° + k{\cdot}180°, k \in \mathbb{Z}$

$\therefore\ \theta = 57,2° + k{\cdot}180°$

1.5 $4 + \tan \frac{\theta}{3} = -2,12$

$\therefore\ \tan \frac{\theta}{3} = -6,12$

$\therefore\ \frac{\theta}{3} = -80,7° + k{\cdot}180°, k \in \mathbb{Z}$

$\therefore\ \theta = -242,2° + k{\cdot}540°, k \in \mathbb{Z}$

2. $2\cos(\theta - 20°) = 0,632$

$\cos(\theta - 20°) = 0,316$

$\therefore\ \theta - 20° = \pm 71,6° + k{\cdot}360°$

$\therefore\ \theta = 91,6° + k{\cdot}360°$ or $\theta = -51,6° + k{\cdot}360°$

Solution set: $\theta \in \{-268,4°; -51,6°; 91,6°\}$

3. $-5\tan\theta = -5,05$

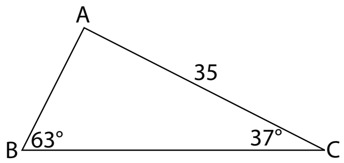
$\tan\theta = 1,01$

$\therefore\ \theta = 45,3° + k.180°, k \in \mathbb{Z}$

Solution Set: $\theta \in \{45,3°; 225,3°\}$

Activity 10

1.



$\frac{AB}{\sin C} = \frac{AC}{\sin B}$ (sine Rule)

$\therefore\ AB = \frac{35}{\sin 63°}{\cdot}\sin 37° = 23,6$

$\therefore AB = 23,6$

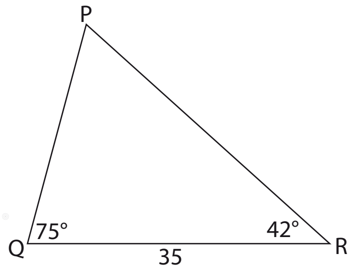
$\hat{A} = 80°$ ($\angle$ sum in $\triangle$)

$\therefore\ \frac{BC}{\sin A} = \frac{AC}{\sin B}$ (Sine Rule)

$\therefore\ BC = \frac{35}{\sin 63°}{\cdot}\sin 80°$

$BC = 38,7°$

2.



$\hat{P} = 63°$ (∠ sum in △)

$\frac{PR}{\sin Q} = \frac{QR}{\sin P}$ (Sine Rule)

$PR = \frac{35}{\sin 63°} \cdot \sin 75°$

$PR = 37{,}9$

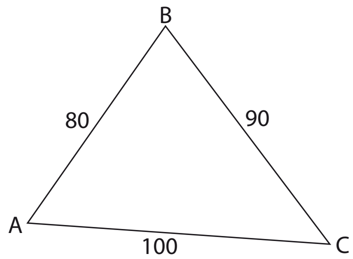
$\frac{PQ}{\sin 42°} = \frac{35}{\sin 63°}$ (Sine Rule)

$PQ = \frac{35}{\sin 63°} \cdot \sin 42°$

$PQ = 26{,}3$

Activity 11

1.



$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos B$

$(100)^2 = (80)^2 + (90)^2 - 2(80)(90) \cdot \cos B$

$\therefore \frac{(100)^2 - (80)^2 - (90)^2}{-2(80)(90)} = \cos B$

$0{,}3125 = \cos B$

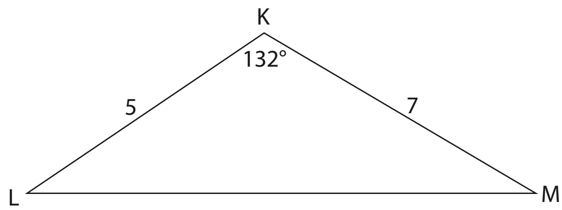
$71{,}8° = \hat{B}$

$\frac{\sin C}{80} = \frac{\sin B}{100}$ (Sine Rule: $\hat{B} = 71{,}8°$)

$\therefore \sin C = 0{,}75\ldots$

$\therefore \hat{C} = 49{,}5°$ $\therefore \hat{A} = 58{,}7°$ (∠ in sum △)

2.



$LM^2 = KL^2 + KM^2 - 2KL.KM.\cos K$

$LM^2 = (5)^2 + (7)^2 - 2(5)(7)\cos 132°$

$LM^2 = 120{,}8\ldots$

$LM = 10{,}99\ldots = 11{,}0$

$\frac{\sin \overline{M}}{5} = \frac{\sin 132°}{11}$

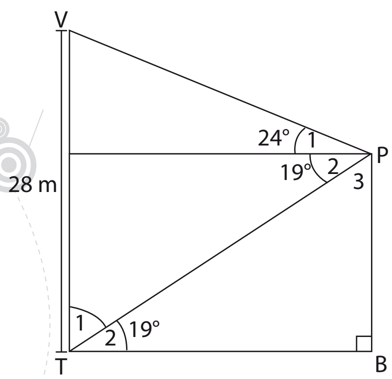
$\therefore \sin \overline{M} = \frac{\sin 132°}{11} \cdot 5$

$\sin \overline{M} = 0{,}33\ldots$

$\therefore \hat{M} = 19{,}7°$ $\therefore \hat{L} = 28{,}3°$ (∠ sum in △)

Activity 12

1.



$\hat{T}_1 = 71°$ (VT⊥TB)

$\hat{V} = 66°$ (∠ sum in △PVT)

In △VTP: $\frac{PT}{\sin V} = \frac{TV}{\sin(\hat{P}_1 + \hat{P}_2)}$

$PT = \frac{28}{\sin(43°)} \cdot \sin 66°$

PT = 37,5 metres

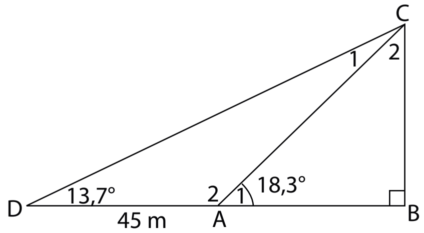
LIBERTY LIFE

$\cos 19° = \frac{TB}{37{,}5}$ (Working in right-angled $\triangle TBP$)

$35{,}5m = TB$

Building is 35,5 metres from the tower.

2.



$A_2 = 161{,}7°$ (str. angle)

$\hat{C}_1 = 4{,}6°$ ($\angle$ sum in $\triangle$)

$\frac{AC}{\sin D} = \frac{AD}{\sin C_1}$ (sin rule in $\triangle ADC$)

$\frac{AC}{\sin 13{,}7°} = \frac{45}{\sin 4{,}6°}$

$AC = 132{,}9$ metres

In $\triangle ABC$: $\frac{AB}{AC} = \cos A_1$

$\frac{AB}{132{,}9} = \cos 18{,}3$ $\therefore AB = 126{,}2$ metres

$\frac{BC}{AB} = \tan 18{,}3°$ $\therefore BC = 41{,}7$ metres

Height of Tree = 41,7 m Width of River = 126,2 metres

3.1 $BAC = 108°$ (str $\angle$) $\therefore BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos 108°$

$BC^2 = (4{,}5)^2 + (5{,}6)^2 - 2(4{,}5)(5{,}6) \cdot \cos 108°$

$BC = 8{,}2$ km

3.2 $\frac{\sin C}{4{,}5} = \frac{\sin 108°}{8{,}2}$

$\sin C = 0{,}52\ldots$

$\hat{C} = 31{,}5°$

3.3 Area $\triangle ABC = \frac{1}{2}(4{,}5)(5{,}6)\sin 108°$

$= 11{,}98$ sq km

Activity 13

1.1.1 $\sin 161° = \sin 19° = t$

1.1.2 $\cos 71° = \sin 19° = t$

1.1.3 $\sin 341° = -\sin 19° = -t$

1.1.4 $\cos 19° = \sqrt{1 - \sin^2 19°} = \sqrt{1 - t^2}$

1.1.5 $\tan 71° = \frac{\sin 71°}{\cos 71°} = \frac{\cos 19°}{\sin 19°} = \frac{\sqrt{1 - t^2}}{t}$

1.2 $\frac{(-\cos 45°) \cdot \tan 60° \cdot (-\sin 45°)}{\cos 60°} = \left(\frac{-1}{\sqrt{2}}\right)(\sqrt{3})\left(\frac{-1}{\sqrt{2}}\right) \div \frac{1}{2}$

$= \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

1.3 $\frac{\sin 80° \cdot (-\sin x)(-\sin x)}{(\sin x)(\cos x)(-\tan x) \cdot \cos 10°}$

$= \frac{\cos 10°(\sin^2 x)}{\cos 10° \cdot (\sin x)(\cos x) \cdot \frac{-\sin x}{\cos x}}$

$= -1$

1.4 LHS: $\dfrac{\sin^2\beta + \cos\beta(1+\cos\beta)}{(1+\cos\beta)(\sin\beta)}$ RHS: $\dfrac{1}{\sin\beta}$

$= \dfrac{\sin^2\beta + \cos\beta + \cos^2\beta}{(1+\cos\beta)(\sin\beta)}$

$= \dfrac{(1+\cos\beta)}{(1+\cos\beta)\sin\beta}$

$= \dfrac{1}{\sin\beta}$

$\therefore$ LHS = RHS

1.5.1 $\tan A = -0{,}758$

$A = -37{,}2° + k\cdot 180°, k \in \mathbb{Z}$

1.5.2 $\cos A + 0{,}32 = 1$

$\cos A = 1 - 0{,}32 = 0{,}68$

$A = 47{,}2 + k\cdot 360°$ or $A = -47{,}2° + k\cdot 360°, k \in \mathbb{Z}$

1.5.3 $\sin 2A = 0{,}72$

$2A = 46,\ldots° + k\cdot 360°$ or $2A = 180° - (46,\ldots°) + k\cdot 360°, k \in \mathbb{Z}$

$A = 23{,}03° + k\cdot 180°$ $2A = 133{,}9\ldots° + k\cdot 360°$

$A = 66{,}97° + k\cdot 180°$

2.1.1 $\frac{3}{2}\tan^2(-30°) - \frac{3}{2}\cos 300° - 2\sin^2(-1\,035)$

$= \frac{3}{2}\tan^2(30°) - \frac{3}{2}\cos 60° - 2\sin^2(1\,035)$

$= \frac{3}{2}\left(\frac{1}{\sqrt{3}}\right)^2 - \frac{3}{2}\left(\frac{1}{2}\right) - 2\sin^2(315°)$

$= \frac{3}{2}\left(\frac{1}{3}\right) - \frac{3}{4} - 2\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} - \frac{3}{4} - 1 = \frac{-5}{4}$

2.1.2 $\sin 303°\cdot\cos 213° - \frac{1}{\tan 123°}\cdot\cos 33°\cdot\cos(-57°)$

$= (-\sin 57°)(-\cos 33°) - \frac{1}{(-\tan 57°)}\cdot\cos 33°\cdot(\cos 57°)$

$= \sin 57°\cdot\sin 57° + \frac{1}{\tan 57°}\cdot\sin 57°\cdot\cos 57°$

$= \sin^2 57° + \frac{\cos 57°}{\sin 57°}\cdot\sin 57°\cdot\cos 57°$

$= \sin^2 57° + \cos^2 57° = 1$

2.2.1 LHS: $\left(\dfrac{\sin A}{\cos A} + 1\right)\left(\dfrac{1}{\frac{\sin A}{\cos A}} + 1\right)$

$= \left(\dfrac{\sin A + \cos A}{\cos A}\right)\left(\dfrac{\cos A}{\sin A} + 1\right)$

$= \left(\dfrac{\sin A + \cos A}{\cos A}\right)\left(\dfrac{\cos A + \sin A}{\sin A}\right)$

$= \dfrac{\sin^2 A + 2\sin A\cos A + \cos^2 A}{\sin A\cos A}$

$= \dfrac{1 + 2\sin A\cos A}{\sin A\cos A}$

$= \dfrac{1}{\sin A\cos A} + \dfrac{2\sin A\cos A}{\sin A\cos A}$

$= \dfrac{1}{\sin A\cos A} + 2 = \text{RHS}$

2.2.2 $\dfrac{\cos A}{1+\sin A} + \dfrac{1+\sin A}{\cos A}$

$= \dfrac{\cos^2 A + (1+\sin A)^2}{(1+\sin A)\cos A}$

$= \dfrac{\cos^2 A + 1 + 2\sin A + \sin^2 A}{(1+\sin A)\cos A}$

$= \dfrac{1 + 1 + 2\sin A}{(1+\sin A)(\cos A)} = \dfrac{2(1+\sin A)}{(1+\sin A)\cos A} = \dfrac{2}{\cos A} = \text{RHS}$

2.2.3 $\frac{1-\sin\theta}{1+\sin\theta} = \text{LHS}$ RHS: $\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$

$$= \frac{(1-3\sin\theta)^2}{\cos^2\theta}$$

$$= \frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{1-\sin\theta}{1+\sin\theta}$$

LHS = RHS

2.2.4 LHS $\left[\frac{1}{-\sin\theta} - \frac{1}{\cos\theta}\right] \div \left[\tan\theta - \frac{1}{-\tan\theta}\right]$

$$= \frac{-\cos\theta - \sin\theta}{\sin\theta\cos\theta} \div \left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right]$$

$$= \frac{-(\cos\theta + \sin\theta)}{\sin\theta\cos\theta} \times \frac{\cos\theta.\sin\theta}{\sin^2\theta + \cos^2\theta}$$

$$= -\cos\theta - \sin\theta = \text{RHS}$$

2.3 $\cos(2\beta - 30°) = -0{,}378$

$2\beta - 30° = 112{,}2\ldots° + k\cdot360°$ or $2\beta - 30° = -112{,}2\ldots + k\cdot360°$

$2\beta = 142{,}2\ldots + k\cdot360°$ $2\beta = -82{,}2\ldots + k\cdot360°$

$\beta = 71{,}1° + k\cdot180°$ $\beta = -41{,}1° + k\cdot180°, k \in \mathbb{Z}$

3.1 $\hat{F}_2 = 57{,}2°$

$\hat{H}_1 = 8{,}5°$ ($\angle$ sum in $\triangle$HTF)

In $\triangle$HTF: $\frac{HF}{\sin 114{,}3} = \frac{95}{\sin 8{,}5}$

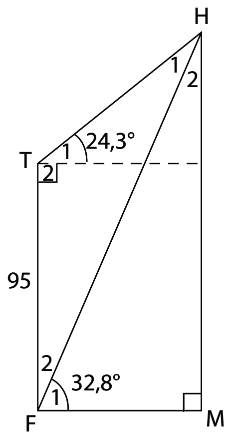
$HF = 585{,}77\ldots$

In $\triangle$FHM: $\frac{HM}{HF} = \sin F_1$

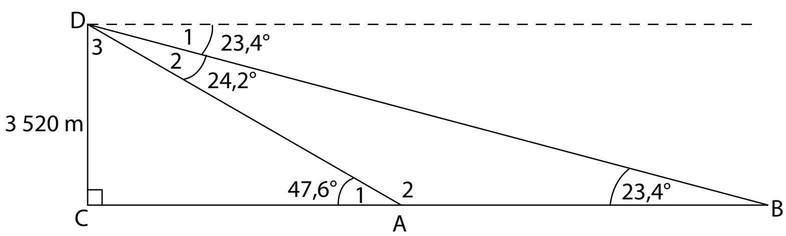
$HM = (\sin 32{,}8)(585{,}77\ldots)$

$= 317{,}3$ metres

Hill is 317,3 metres high.



3.2



In $\triangle$DCA: $\frac{3\,520}{AD} = \sin 47{,}6°$

$AD = 4\,766{,}7\ldots$

In $\triangle$DAB: $\frac{AB}{\sin 24{,}2°} = \frac{4\,766{,}7\ldots}{\sin 23{,}4°}$

$AB = 4\,920{,}0$ m

$AB = 4{,}92$ km