LESSON 2

SEQUENCES & SERIES

In mathematics you have already had some experience of working with number sequences and number patterns. In grade 11 you learnt about *quadratic* or *second difference* sequences. In grade 12 you will learn more about two specific types of sequences which both have quite unique properties. These are **arithmetic sequences** and **geometric sequences.** Let's start with the arithmetic sequences (which you have probably seen before).

Arithmetic sequences

Here is an example of an arithmetic sequence: 2; 5; 8; 11; …

There are a couple of important things that you should notice. Most importantly, you should notice that there is a difference of 3 between each number in the sequence. This enables you to predict the next number in the sequence at any point in the sequence. All arithmetic sequences (also known as arithmetic progressions or A.P.'s) are sequences in which the difference between any two successive terms is a constant. We say that these sequences have a **common difference** between terms.

You should also realise that the sequence can continue indefinitely. If we know the starting point of the sequence, and the common difference between terms, then we can generate the unique sequence and start to make mathematical predictions, like, "finding the value of the 10th term " or, "finding the sum of the first 15 terms" etc.

Let's introduce the notation that will be used in this section.

We generally refer to the first value (term) of a sequence by using the letter *a*, and the common difference by the letter *d*.

Generate a sequence with $a = -10$ and $d = 4$:

$$-10; \xrightarrow{+4} -6; \xrightarrow{+4} -2; \xrightarrow{+4} 2; \xrightarrow{+4} 6; \quad \ldots$$

We notice that 6 is the value of the 5th term in the sequence. We represent this as $T_5 = 6$.

Now we can refer to the first term as *a* or as T_1.

$T_2 - T_1 = -6 - (-10) = 4$
$T_3 - T_2 = -2 - (-6) = 4$
$T_4 - T_3 = 2 - (-2) = 4$
$T_5 - T_4 = 6 - (2) = 4$

Common difference ($d = 4$)

Thus $T_1 = a$

$T_2 = a + d$

$T_3 = (a + d) + d = a + 2d$

$T_4 = (a + 2d) + d = a + 3d$

So $T_{20} = a + 19d$ and by generalising this *rule*, we generate the formula

$\boxed{T_n = a + (n-1)d}$ where *n* is the number of terms, which is always a positive, whole number.

In order to confirm that a sequence is arithmetic, all we need to do is deduce that there is a common difference between terms.

So $\boxed{d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}}$

Example

Examples

Find the 12th term of the sequence: 4; 7; 10; 13; …

$a = 4$ and $d = T_2 - T_1 = 7 - 4 = 3$. (Always check to see that this difference is the same **for all** terms in the sequence.)
$n = 12$

Use the formula $T_n = a + (n-1)d$

$\therefore T_{12} = a + 11d$

$T_{12} = 4 + 11(3)$

$T_{12} = 37$

This means that the 12th term has a value of 37.

Finding the *n*th term of the sequence

What does this mean?

It is possible to generate a formula where all we do is substitute a value for n, the position of the term in the sequence, in order to find the value of the term.

e.g. we are given the *n*th term $T_n = 5 - 2n$. So, to find the value of the 10th term all we do is replace *n* by 10. $\therefore T_{10} = 5 - 2(10) = -15$.

We can also find a and d, but in order to do this we need to first generate a few of the terms in this sequence.

Sub $n = 1$: $\therefore T_1 = 5 - 2(1) = 3 \rightarrow$ Now we know that $a = 3$

Sub $n = 2$: $\therefore T_2 = 5 - 2(2) = 1$

Sub $n = 3$: $\therefore T_2 = 5 - 2(3) = -1$

$d = T_2 - T_1 = T_3 - T_2 = -2$

Although it is possible to determine the value of *d*, the common difference, once we have the first two terms, it is always a good idea to check once we have at least 3 terms in order to be certain! A pattern can only emerge once we have 3 or more terms.

e.g. Find the *n*th term of the sequence: 7; –1; –9; –17; …

$a = 7$ and $d = -1 - (7) = -8$

To generate the nth term formula all we do is substitute these values into the general formula

$T_n = a + (n-1)d \therefore T_n = 7 + (n-1)(-8)$

$T_n = 7 - 8n + 8$

$T_n = 15 - 8n$

Hint: With an arithmetic sequence, everything hinges on the values of *a* and *d*. Once you have these values you can find anything…

We can use this *n*th term (general term) formula to generate the 5th term of the sequence:

$T_n = 15 - 8n \therefore T_5 = 15 - 8(5) = -25$

which we can verify by taking the original sequence 7; –1; –9; –17; … $\xrightarrow{-8}$ –25

and 'adding' the common difference to find the next term.

e.g. Find the first term and the common difference of the sequence where $T_n = 6 - 2n$

$T_n = 6 - 2n \therefore T_1 = 6 - 2(1) = 6 - 2 = 4$

$T_2 = 6 - 2(2) = 6 - 4 = 2$

$T_3 = 6 - 2(3) = 6 - 6 = 0$

Thus: $T_1 = 4$ and $d = -2$.

e.g. Which term in the sequence 6; 13; 20; … is equal to 76?

Here we need to find the value of *n*, the number of terms (or the position) in the sequence, to give a value of 76. So $T_n = 76$.

From the sequence: $a = 6$; $d = 7$.

Now all we do is substitute these values into the formula and solve for our unknown, *n*.

$$T_n = a + (n - 1)d \therefore 76 = 6 + (n - 1)(7)$$

$$76 = 6 + 7n - 7$$

$$76 = -1 + 7n$$

$$77 = 7n$$

$$n = 11$$

i.e. the 11^{th} term has a value of 76 or $T_{11} = 76$.

e.g. Determine the arithmetic sequence in which the 21^{st} term is 170 and the 5^{th} term is 122.

This is a particularly important example and the skill used to answer this question is used for other types of sequences as well.

When we are given two 'pieces' of information this usually means in maths that we are going to have to solve simultaneously, especially if we need to find the values of two different variables. Earlier we noticed that everything in arithmetic sequences hinges on the values of *a* and *d*, which is exactly what we need to solve for here.

Hint: Always start by writing your equations in terms of *a* and *d*.

So… $T_{21} = a + 20d$ and $T_5 = a + 4d$

$170 = a + 20d$ $\quad$ $122 = a + 4d$

Now, we can solve simultaneously, in any way that we wish to, but it is usually easier to solve by elimination. All we do is subtract the one equation from the other:

$$170 = a + 20d$$

$$122 = a + 4d$$

$$\therefore 48 = 16d \quad \therefore d = 3$$

$$\text{from } 122 = a + 4d$$

$$\therefore 122 = a + 4(3)$$

$$\therefore a = 110$$

Now we are ready to answer the original question… "find the arithmetic sequence".

All we really need to do is generate the first three terms of the sequence.

So: 110; 113; 116; …

Start with a

Keep adding d to generate successive terms in the sequence.

Activity

Activity 1

1. The sixth term of an arithmetic sequence is 17 and the tenth term is 33. Determine the first term and the common difference.

2. x; $2x + 1$; 11 are three consecutive terms of an arithmetic sequence. Calculate:

   2.1 x

   2.2 the 30th term

3. The first term of an arithmetic sequence is -3 and the third term is 3. Determine:

   3.1 the value of the 25th term of the sequence.

   3.2 which term of the sequence will be equal to 57?

4. Show that the general term of the sequence $\frac{-2}{5}; \frac{1}{10}; \frac{4}{15}; \frac{7}{20}; \frac{10}{25}; \ldots$ can be written as $T_n = \frac{3n - 5}{5n}$

Sigma notation: $\sum$

The Greek letter $\sum$(sigma) is used to indicate the sum of a sequence of numbers. In this section we will learn how to use and interpret sigma notation – which is a very useful and convenient way of expressing the sums of sequences.

e.g. $\sum_{i=1}^{5}(2i - 3)$ means the sum of all the terms from 1 to 5 of the sequence with the general term $(2i - 3)$.

There is usually a link between the variable under the sigma sign (i in this case) and the variable in the general term. We start by replacing i 1, and increment (increase) by 1 each time until we reach the end value, which is 5 here. All the terms must then be added together.

$$\sum_{i=1}^{5}(2i-3) = [(2(1)-3)] + [(2(2)-3)] + [(2(3)-3)] + [(2(4)-3)] + [(2(5)-3)]$$
$$= -1 + 1 + 3 + 5 + 7$$
$$= 15$$

An important trick! Determining the number of terms from sigma notation.

$\sum_{i=15}^{120}(2i-3)$ This sum is quite interesting since i starts at 15. This means that we substitute 15 in place of i in the general term in order to determine the value of the first term. So $T_1 = 2(15) - 3 = 27$.

$\boxed{(120-15)+1}$

$\sum_{i=15}^{120} = T15 + T16 + Tn + ... + T120$ Careful consideration will show that this is 106 terms

To find n, the number of terms, in this series all we do is subtract 15 from 120 (the end value) and **then add 1**… So the number of terms is 106. (Top minus bottom plus 1). Simple!

Notation continued:

e.g. Given $\sum_{i=1}^{7}(4+ti) = 112$, find the value of t.

$$[4+t(1)] + [4+t(2)] + [4+t(3)] + [4+t(4)] + [4+t(5)] + [4+t(6)] + [4+t(7)] = 112$$
$$(4+t) + (4+2t) + (4+3t) + (4+4t) + (4+5t) + (4+6t) + (4+7t) = 112$$
$$28t + 28 = 112$$
$$28t = 84$$
$$t = 3$$

Arithmetic Series

When the terms of an arithmetic sequence are added, then the sequence is known as an arithmetic series.

We can prove that S_n, the sum of n terms, can be calculated using the following formulae:

$S_n = \frac{n}{2}(a + l)$ where l, the last term of the sequence, is equivalent to

$T_n = a + (n-1)d$

But if $l = a + (n-1)d$ then $S_n = \frac{n}{2}[(2a + (n-1)d]$.

Both of these formulae can be used to evaluate the sum of an arithmetic series although you may prefer to use one over the other depending on what information is given in your question.

e.g. Find the sum of the first 20 terms of the arithmetic series (also known as A.P. or arithmetic progression)

$(-16) + (-12) + (-8) + \ldots$

$a = -16$, $d = (-12) - (-16) = 4$, $n = 20$

$S_n = \frac{n}{2}(2a + (n-1)d)$ $\quad\therefore S_{20} = \frac{20}{2}[(2(-16) + (20-1)4)]$

$$\therefore S_{20} = 10\,(-32 + 19(4))$$

$$\therefore S_{20} = 10\,(-32 + 76)$$

$$\therefore S_{20} = 10(44) = 440$$

e.g. Find the sum of the arithmetic series: $12 + 7 + 2 + \ldots + (-43)$

Thus we know the first and the last terms but we do not know how many terms must be added. So, we must start by finding *n*. (Remember we did this in an earlier example.)

$d = 7 - 12 = -5$, $a = 12$ and $T_n = -43$

$$a + (n-1)d = -43$$

$$\therefore 12 + (n-1)(-5) = -43$$

$$\therefore 12 - 5n + 5 = -43$$

$$\therefore -5n = -43 - 12 - 5$$

$$\therefore -5n = -60$$

$$\therefore n = 12$$

Now, we can determine the sum of the first 12 terms of the series, using either of the two formulae.

$S_n = \frac{n}{2}\,[(2a + (n-1)d)]$	or $S_n = \frac{n}{2}(a + l)$
$S_{12} = \frac{12}{2}\,[2(12) + (11)(-5)]$	$\therefore S_{12} = \frac{12}{2}\,(12 + (-43))$
$S_{12} = 6(24 - 55)$	$\therefore S_{12} = 6(-31)$
$S_{12} = 6(-31) = -186$	$\therefore S_{12} = -186$

This above example illustrates that both formulae work equally well!

Now for something a bit more tricky…

When we solve for *n*, the number of terms to give an arithmetic sum, we will always encounter a quadratic equation. This means that we will need to use our knowledge of quadratics and the quadratic formula to help us here.

Also remember that *n*, the number of terms, is a positive whole number. Any values that are not whole numbers must be rejected as possible solutions.

e.g. The sum of $(-8) + (-2) + (4) + \ldots$ is 1600. How many terms are there?

$a = -8$, $d = (-2) - (-8) = 6$ and $S_n = 1600$

$$\frac{n}{2}[2a + (n-1)d] = 1600$$

$\therefore \frac{n}{2}\,[2(-8) + (n-1)(6)] = 1600$ Multiply both sides by 2 to 'remove' fractions

$$\therefore n(-16 + 6n - 6) = 3200$$

$$\therefore n(6n - 22) = 3200$$

$\therefore 6n^2 - 22n - 3200 = 0$ Divide through by the common factor of 2

$$\therefore 3n^2 - 11n - 1600 = 0$$

$\therefore (3n + 64)(n - 25) = 0$ It may be advisable to use the quadratic formula at this stage.

$\therefore n = -\frac{64}{3}$ or $n = 25$

We must reject this solution.

So, the number of terms is 25.

Now earlier in this chapter we encountered a problem where we were given two pieces of 'information'. This is not uncommon and hopefully you remember that the key to this type of problem is to always write equations in terms of a's and d's… so you only have two unknowns, and so you can solve simultaneously.

e.g. The 8th term of an arithmetic series is 16 and the sum of the first 10 terms is 210. Find the sum of 15 terms.

$T_n = a + (n - 1)d$ and $S_n = \frac{n}{2}[(2a + (n - 1)d]$

$T_8 = 16 = a + 7d$ $\qquad S_{10} = 210 = \frac{10}{2}[(2a + (9)d)]$

$\therefore\ a = 16 - 7d \quad \ldots(1)$ $\qquad 210 = 5(2a + 9d)$ Divide both sides by 5

$42 = 2a + 9d \quad \ldots(2)$

Now we can substitute $16 - 7d$ in place of a in equation 2, so that we are only solving for 1 unknown…

From (2): $42 = 2(16 - 7d) + 9d$

$42 = 32 - 14d + 9d$

$\therefore 10 = -5d$

$\therefore d = -2$

$\therefore a = 16 - 7(-2) = 30$

Once we have a and d, anything can be determined…

So, from $S_n = \frac{n}{2}(2a + (n - 1)d) \rightarrow S_{15} = \frac{15}{2}(2(30) + 14(-2))$

$S_{15} = \frac{15}{2}(60 - 28)$

$S_{15} = 240$

Activity

Activity 2

1. Determine the value of n if $\sum_{p=1}^{n} 2p - 3 = 80$.

2. Given: $6 + 1 - 4 - 9 - \ldots - 239$. Evaluate:

   2.1 The number of terms in the arithmetic sequence above.

   2.2 The sum of the series.

3. Cara works as a newspaper delivery agent and initially earns R15 000 in her first year. Thereafter her salary increases by R1 500 per year. Her expenses are R13 000 during the first year, and then they increase by R900 in each subsequent year. Use a formula to determine how long it would take her to save R21 000, assuming that the money saved each year is not deposited into an account (so no interest is added).

Geometric Sequences

A geometric sequence (or progression) is formed when each term is multiplied by the same number to get to the next term. We call this number the **common ratio**.

Here is an example of a geometric sequence: 3; 6; 12; 24; 48; …

You should notice that each term is doubled in the sequence. So the number that we are multiplying by each time is 2.

The notation used is very similar to before except that we do not have a common difference but rather a common ratio or r (i.e. $r = 2$). We still refer to the first value (term) of the sequence by using the letter a.

Generate a sequence with $a = -81$ and $r = \frac{1}{3}$: (Note: multiplying by $\frac{1}{3}$ is equivalent to dividing by 3)

$$-81 \xrightarrow{\times\frac{1}{3}} -27 \rightarrow -9 \rightarrow -3 \rightarrow -1 \rightarrow -\tfrac{1}{3}$$

$-81;\ -27;\ -9;\ -3;\ -1;\ -\frac{1}{3};\ \ldots$

> If we had been given $r = -\frac{1}{3}$ can you see why the signs of the terms in the sequence would have alternated? i.e. Each term would have changed sign.

Since we generate successive terms by multiplying by the common ratio, we can determine the value of the common ratio by dividing successive terms.

$$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{-27}{-81} = \frac{1}{3} \\ \frac{T_3}{T_2} &= \frac{-9}{-27} = \frac{1}{3} \\ \frac{T_4}{T_3} &= \frac{-3}{-9} = \frac{1}{3} \end{aligned}\right\rangle \text{Common ratio } (r)$$

Thus $T_1 = a$

$T_2 = a \times r = ar$

$T_3 = ar \times r = ar^2$

$T_4 = ar^2 \times r = ar^3$

So $T_{20} = ar^{19}$ and by generalising this *rule*, we generate the formula

$T_n = ar^{n-1}$ where n is the number of terms, which is always a positive, whole number.

In order to tell that a sequence is geometric, all we need to do is deduce that there is a common ratio between terms.

So $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots = \frac{T_n}{T_{n-1}}$

Example

Examples

Find the 8[th] term in the G.P. (geometric progression) 32; 16; 8; …

$a = 32;\ r = \frac{16}{32} = \frac{1}{2}$

$\therefore\ T_n = ar^{\,n-1} \rightarrow T_8 = ar^7$

$\therefore T_8 = 32\left(\frac{1}{2}\right)^7$

$\therefore T_8 = \frac{1}{4}$ or 0,25

We can obtain the solution directly if a calculator is used.

e.g. Which term in the sequence 5; 15; 45; … has a value of 3645?

$a = 5;\ r = \frac{15}{5} = 3;\ T_n = 3645$

$ar^{n-1} = 3645$ $\quad\therefore\ 5(3)^{n-1} = 3645$ (Divide both sides by 5)

$\therefore\ (3)^{n-1} = 729$

$\therefore\ (3)^{n-1} = 3^6$

$\therefore\ n - 1 = 6$

$\therefore\ n = 7$

e.g. Find a if the 7[th] term of a geometric sequence is 1024 and $r = 4$.

$T_7 = 1024 = ar^6$ Now we substitute $r = 4$, and solve for a.

$1024 = a(4)^6$

$\therefore\ 1024 = a(4096)$

$\therefore\ \frac{1024}{4096} = \frac{1}{4} = a$

The only other unknown that we need to practice solving for is r. Here a good knowledge of exponents and powers is quite useful.

e.g. Find r if the 9[th] term of the G.P. is $\frac{3}{128}$ and the first term is 6.

We know that $T_9 = ar^8 = \frac{3}{128}$ where $a = 6$.

$6r^8 = \frac{3}{128}$

$r^8 = \frac{3}{128} \times \frac{1}{6}$ Divide both sides by 6 (multiply by $\frac{1}{6}$)

$r^8 = \frac{1}{256}$ Simplify the fraction… use a calculator

$r^8 = \left(\frac{1}{2}\right)^8$

or $r = \sqrt[8]{\frac{1}{256}}$

$\therefore r = \pm\frac{1}{2}$

At this stage we can simplify by either recognising that both bases can be written with the same power or by using our calculator and taking the root…

Now let's try an example where we are given two pieces of information. Just like earlier, when we worked with arithmetic sequences, our goal here is to express all unknowns in terms of a and r. Then, when we have two equations involving two unknowns, we can **solve simultaneously**. This particular 'trick' comes up often in this section.

e.g. If the 2nd term of a geometric sequence is –3 and the 5th term is 24, find the first 3 terms of the geometric sequence.

$T_2 = -3$

$\therefore \quad ar = -3$

$\therefore \quad a = \frac{-3}{r}$

Always solve for *a*, and substitute the simpler equation into the more complicated equation.

$T_5 = 24$

$\therefore ar^4 = 24$

At this stage we form an equation in terms of one variable only… so we can solve easily.

$\therefore \frac{-3}{r} \times r^4 = 24$

$-3r^3 = 24$ (Divide by –3)

$\therefore r^3 = -8$

Now we cube root both sides in order to solve for r: $r = -2$

When r is a negative we get a special type of sequence, called an alternating sequence. This is because each term in the sequence changes sign since multiplying by a negative has that effect!

Now we are ready to solve for a and of course, the first 3 terms.

From $a = \frac{-3}{r} \rightarrow a = \frac{-3}{-2} = \frac{3}{2}$

$\therefore$ G.P. is $\frac{3}{2}; \xrightarrow{\times -2} -3; \xrightarrow{} 6; \ \ldots$

Activity 3

Activity

1. The third term of a geometric sequence is 4 and the 6th term is $\frac{32}{27}$. Find the nth term.

2. $T_n = 3r^{n-1}$, $r > 0$, is the nth term of a geometric sequence. If the 3rd term is 48, determine r.

3. 5; x; y is an arithmetic sequence and $x;y;81$ is a geometric sequence. All terms in the sequence are integers. Calculate the values of x and y.

4. You start working for a company as a sales rep. and obtain a basic salary of R2 000 in your first month employed. Thereafter your salary decreases each month by 10% of the previous months salary. If you work for 1 entire year, will you earn more than R1 000 in December (the 12th month) assuming that you start working in January, and earn your first salary at the end of this month?

Geometric series

When the terms of a geometric sequence are added, then the sequence is known as a geometric series.

We can prove that S_n, the sum of n terms, can be calculated using either of the two formulae:

$S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{a(1 - r^n)}{1 - r}$

e.g. Find the sum of the first 8 terms of the sequence 5 + 15 + 75 + …

$r = \frac{T_2}{T_1} = \frac{15}{5} = 3;\ a = 5;$

$\therefore\ S_n = \frac{a(r^n - 1)}{r - 1}\quad \therefore\quad S_8 = \frac{5(3^8 - 1)}{5 - 1}$

$S_8 = \frac{5(3^8 - 1)}{4}$

$S_8 = \frac{5(6561 - 1)}{4} = \frac{5(6560)}{4} = 8200$

Use a calculator where necessary.

e.g. Find the sum of the first 10 terms of the series 3 – 12 + 48 + …

Here you should notice that the series is an alternating sequence, so r is negative.

$r = \frac{T_2}{T_1} = \frac{-12}{3} = -4\ ;\ a = 3$

$\therefore\ S_{10} = \frac{3((-4)^{10} - 1)}{-4 - 1}$

$\therefore\ S_{10} = \frac{3(1048576 - 1)}{-5}$

$\therefore S_{10} = -629145$

Now let's try something a bit trickier called the **sum to infinity**. We only ever determine the sum to infinity for geometric sequences that converge. We say that a sequence converges when $-1 < r < 1$. So if $r = 2$, for example, the series increases as n increases. i.e. 2^n becomes infinitely large.

But if $r = \frac{1}{2}$, then r^n decreases as n increases: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$; $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$; $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$

Thus $\lim_{n \to \infty} r^n = 0$ when $-1 < r < 1$.

It can be shown that the sum to infinity: $S_\infty = \frac{a}{1 - r}$ *if* $-1 < r < 1$

e.g. Find the S_∞ for the sequence: $8; \frac{16}{3}; \frac{32}{9}; \frac{64}{27}; \ldots$

This appears absurd at first… that the sum of infinitely many terms of a geometric sequence can converge to a finite number. But this can be proved, so long as $-1 < r < 1$. So we must start our solution by determining the value of r.

$r = \frac{T_2}{T_1} = \frac{\frac{16}{3}}{8} = \frac{16}{3} \times \frac{1}{8} = \frac{2}{3};$ Remember that when dividing through by fractions, always invert and multiply.

$a = 8$

$S_\infty = \frac{a}{1 - r}$ The value of r assures us that the series will converge, since $-1 < \frac{2}{3} < 1$

$S_\infty = \frac{8}{1 - \frac{2}{3}} = \frac{8}{\frac{1}{3}}$ Once again, remember to 'invert and multiply'.

$S_\infty = 8 \times \frac{3}{1} = 24$

Activity 4

Activity

1. The first term of a geometric sequence is $\frac{2}{3}$ and its fifth term is $\frac{27}{8}$. Determine the sum, rounded to two decimal places, of the first five terms of this sequence. ($r > 0$)

2. Given the geometric series: $6 + 3 + \frac{3}{2} + \frac{3}{4} + \ldots$

   2.1 Find the sum of the first eleven terms of the series.

   2.2 Find the sum to infinity of this series.

3. Determine r if $\sum_{n=1}^{\infty} 2r^{n-1} = 12$.

4. The 'tax man' says that you must pay tax at a flat rate of 20% of your total earnings if you earn more than R50 000 per year, otherwise you must pay tax at a rate of 15%. If you earn R3 500 in the first month of the year, and thereafter for each subsequent month of the year your salary increases by 5% of the previous months salary, how much tax to you pay?

5. The lengths of the radii of circles form an infinite geometric series. The length of the first circle is 10cm. The length of the radius of each of the circles is $\frac{4}{5}$ of the length of the previous circle. Determin the total area of all of the circles formed in this way.

Much of the work on sequences and series relates to finance. You will see in later chapters that the formulae from sequences and series are used to derive the annuities formulae, which forms the bulk of the grade 12 section on the mathematics of finance.

Below is an example of a question where we will solve using sequences and series, but later on in the year we will see how we can also solve using the formulas derived in the finance section.

Example

Example

Samantha deposits R5 000 into a bank account at the end of the year. She then deposits an additional R5 000 at the end of each subsequent year, for an additional 4 years. Interest is added at a rate of 10%p.a. Determine how much money is in the account at the end of the 5th year.

Note: the money deposited initially will be in the bank for a longer period of time ans so it will 'grow' more with interest.

Solution

It is often easier to take each deposit in isolation and 'grow' them to the end of the term.

End of year:	1 ;	2 ;	3 ;	4 ;	5
	5000 ;	5000 ;	5000 ;	5000 ;	5000
	'grows' for 4 more years	'grows' for 3 more years	'grows' for 2 more years	'grows' for 1 more year	no growth. Deposited at the end

Series: $5000(1+\frac{10}{100})^4 + 5000(1+\frac{10}{100})^3 + 5000(1+\frac{10}{100})^2 + 5000(1+\frac{10}{100})^1 + 5000$

Rearrange and simplify: $5000 + 5000(1,1) + 5000(1,1)^2 + 5000(1,1)^3 + 5000(1,1)^4$

This is a geometric series:

$a = 5000 \;;\; r = 1,1 \;;\; n = 5$ (since there are 5 years)

$$\therefore S_5 = \frac{5000[(1,1)^5 - 1]}{1,1-1}$$ From: $S_n = \frac{a(r^n-1)}{r-1}$

$$= R30\ 525,50$$

Let's recap what we have learnt thus far, in the form of a summary comparing arithmetic and geometric sequences and series.

$a =$ 1st term (T_1)

T_n = any term (eg: T_3 is the 3rd term)

n = number of terms in the progression

d = common difference (for A.P.)

r = common ratio (for G.P.)

arithmetic sequence: $a ; a+d ; a+2d ; \ldots$ Common **difference** between terms: $d = T_2 - T_1 = T_3 - T_2$	**geometric sequence:** $a ; ar ; ar^2 ; \ldots$ Common **ratio** between terms: $r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$
General formula to find a term:	
$T_n = a + (n-1)d$	$T_n = ar^{n-1}$
General formula to find the sum of series:	
arithmetic series: $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}(a + l)$ where l is the last term	**geometric series:** $S_n = \frac{a(r^n - 1)}{r - 1}$; $r \neq 1$ **To find the sum to infinity of a CONVERGENT series:** $S_\infty = \frac{a}{1-r}$; $-1 < r < 1$

Activity 5: Mixed past paper questions

Activity

1. The 1st term of an arithmetic sequence is 1 and the 9th term is 15. Calculate the middle term.

2. If $5\frac{1}{3}$ and $40\frac{1}{2}$ are the second and last terms of a geometric progression, find the sum of the first five terms if there are seven terms in the geometric progression.

3. A shrub of height 110 cm is planted. At the end of the first year the shrub is 120 cm tall. Thereafter the growth of the shrub each year is half of its growth in the previous year. Show that the height of the shrub will never exceed 130 cm.

4. The sum of the 4th and the 9th terms of an arithmetic sequence is 0. If the sum of the first 6 terms is 36, determine the 5th term.

5. $S_n = 4n^2 + 1$ represents the sum of the first n terms of a particular series. Determine the value of the 8th term.

6. Calculate: $\sum_{k=2}^{\infty} 8\left(\frac{1}{2}\right)^{k+2}$.

7. The sequence $\frac{1}{3}; x; \frac{16}{3}$ is given. Determine the value of x if the sequence is

7.1 arithmetic.

72 geometric.

8. Consider the geometric sequence: $8(x-2)^2 ; 4(x-2)^3 ; 2(x-2)^4 ;\ x \neq 2$

8.1 Determine the value of x for which the sequence converges.

8.2 Determine the sum to infinity of the series if $x = 2{,}5$.

10. Calculate the value of n, the number of terms, in the equation:
$\sum_{p=1}^{n}(2p - 3) = 120$.

Activity 6

Activity

All the questions in this activity appeared in the DOE Exemplars, Prelim Papers or Additional Exemplars in 2008. The mark allocation has been included for each question. (Usually this section is worth $\pm$ 25–30 marks, which is approximately 20% of the paper.) Also remember that you need to complete 150 marks in 3 hours so, for example, a 10 mark question should take you about 12 minutes to complete. Time yourself when you do this exercise.

1. Determine how may terms the following sequence has:

   $-5; -1; 3; 7; \ldots ; 439$ (3)

   DOE Additional Exemplar 2008 (Q 2.1) **[3]**

2. Consider the following sequence of numbers: 1; 2; 1; 5; 1; 8; 1; 11; …

2.1 What is the 10th term of the above sequence? (2)

2.2 Calculate the sum of the first 50 terms of the sequence. (4)

DOE – Prep Exam 2008 (Q3) **[6]**

3. Consider the following geometric sequence: $81p; 27p^2; 9p^3; 3p^4; \ldots$ $(p \neq 0)$

3.1 Determine the common ratio of the sequence in terms of p. (2)

3.2 For which value of p will the sequence converge? (3)

3.3 Calculate S_∞ if $p = 2$. (3)

DOE Additional Exemplar 2008 (Q2) **[8]**

4. Tebogo and Thembe were investigating the following sequence of numbers:

   2 ; 6 ; 18 ; …

4.1 Tebogo claimed that the fourth term of the sequence is 54. Thembe disagreed and said that the next term is 38. Explain why it is possible that both of them are correct. (4)

4.2 Determine the general term of the sequence in both cases. (7)

4.3 Calculate the 11^{th} term of the sequence according to Thembe's pattern. (2)

4.4 How many terms of Tebogo's pattern will give a sum of 531 440? (3)

DOE Additional Exemplar 2008 (Q3) **[16]**

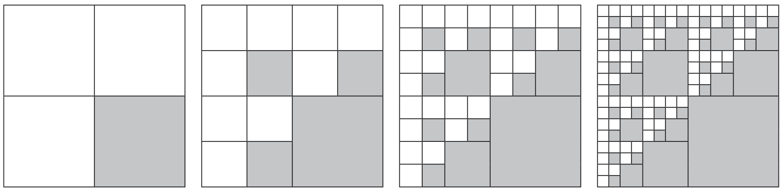
5. Kopano wants to buy soccer boots costing R800, but he only has R290,00. Kopano's uncle Stephen challenges him to do well in his homework for a reward. Uncle Stephen agrees to reward him with 50c on the first day he does well in his homework, R1 on the second day, R2 on the third day, and so on for 10 days.

5.1 Determine the total amount uncle Stephen gives Kopano for 10 days of homework well done. (5)

5.2 Is it worth Kopano's time to accept his uncle's challenge? Substantiate your answer. (2)

DOE – Prep Exam 2008 (Q5) **[7]**

6.



In the patterns above each consecutive pattern has more shaded squares than the previous one. The area of the shaded portion of the first pattern is $\frac{1}{4}$ square units. Assume that the pattern behaves consistently, as shown above.

6.1 The shaded area in Pattern 2 is $\frac{1}{4} + \frac{3}{16}$. Write down the area of the **shaded portions** of Patterns 3 and 4. (4)

6.2 Write down the area of the shaded portion of the n^{th} pattern in sigma notation. (3)

6.3 If the pattern continues without end, what does the area in QUESTION 6.2 approach? (2)

DOE Exemplar 2008 (Q5) **[9]**

Solutions to Activities

Activity 1

1. $T_6 = a + 5d = 17$ .. (1) and $T_{10} = a + 9d = 33$… (2)

from (1) $a = 17 - 5d$ substitute into equation (2):

$17 - 5d + 9d = 33$

$4d = 33 - 17$

$4d = 16 \quad \therefore d = 4$

from $a = 17 - 5d \rightarrow a = 17 - 5(4) = -3$

2. x; $2x + 1$; 11

2.1 $d = T_2 - T_1 = T_3 - T_2$

$\therefore\ 11 - (2x + 1) = (2x + 1) - x$

$\therefore\ -2x + 10 = x + 1$

$\therefore \quad 9 = 3x \quad \therefore\ x = 3$

2.2 $T_1 = x = 3$

$T_2 = 2x + 1 = 2(3) + 11 = 7$

$T_3 = 1 \ \therefore\ a = 3$ and $d = 4$

$T_{30} = a + 29d = 3 + (29)4 = 119$

3. $a = -3$ and $T_3 = a + 2d = 3 \ \therefore\ -3 + 2d = 3 \ \therefore\ d = 3$

3.1 $T_{25} = a + 24d = -3 + 24(3) = 69$

3.2 $T_n = 57 = a + (n - 1)\,d$

$57 = -3 + (n - 1)3 \quad \therefore\ 60 = 3n - 3 \quad \therefore \frac{63}{3} = \mathbf{21} = n$

4. Some sequences can be made up of two different sequences, like we have here. If we separate the terms in the numerator from the terms in the denominator we get two familiar looking sequences:

-2 ; 1 ; 4 ; 7 ; 10 ; … $T_n = 3n - 5$ and 5 ; 10 ; 15 ; 20 ; 25 ; … $T_n = 5n$

Now we just combine the two seqences to get $T_n = \frac{3n-5}{5n}$

Activity 2

1. Given: $\sum_{p=1}^{n} 2p - 3 = 80 \rightarrow$ $\left.\begin{array}{l} T_1 = 2(1) - 3 = -1 \\ T_2 = 2(2) - 3 = 1 \\ T_3 = 2(3) - 3 = 3 \end{array}\right\} a = -1; d = 2$

$S_n = \frac{n}{2}(2a + (n-1)d) = 80$

$\therefore \frac{n}{2}(-2 + (n-1)2) = 80 \;\therefore\; n(2n-4) = 160$

$\therefore 2n^2 - 4n - 160 = 0 \;\; (\div 2)$

$\therefore n^2 - 2n - 80 = 0$

$\therefore (n-10)(n+8) = 0 \;\therefore\; n = 10$ (reject $n = -8$)

2. Given: $6 + 1 - 4 - 9 - \ldots - 239. \rightarrow a = 6; d = -5$

2.1 $a + (n-1)d = -239 \;\; \rightarrow 6 + (n-1)(-5) = -239$

$-5n + 5 = -245 \;\;\therefore\; n = 50$

2.2 $\therefore\; S_{50} = \frac{50}{2}(2(6) + (50-1)(-5)) = 25\,(12 + 49\,(-5)) = -5825$

3. Income: 15000 ; 16500 ; 18000 ; 19500 ; …

Expenses: 13000 ; 13900 ; 14800 ; 15700 ; …

Savings: 2000 ; 2600 ; 3200 ; 3800 ; …

Savings series: $a = 2000 ; d = 600$

$2000 + 2600 + 3200 + \ldots = 21\,000$

$Sn = \frac{n}{2}[2(2000) + (n-1)(600)] = 21\,000$

$\frac{n}{2}[3400 + 600n] = 21\,000$

$1700n + 300n2 = 21\,000 \qquad (\div 100)$

$3n^2 + 17n - 210 = 0$

$(3n + 35)(n - 6) = 0$

$\therefore\; n = \frac{-35}{3}$ (invalid) or $n = 6$

$\therefore$ after 6 years

Activity 3

1. $T_3 = ar^2 = 4$ and $T_6 = ar^5 = \frac{32}{27}$. Solving simultaneously…

from $a = \frac{4}{r^2}$ substitute into T_6: $\therefore\; \frac{4}{r^2} \times r^5 = \frac{32}{27}$

$\therefore 4r^3 = \frac{32}{27}$ $\qquad \therefore\; r^3 = \frac{32}{27} \times \frac{1}{4}$

$\therefore r^3 = \frac{8}{27}$ $\qquad \therefore\; r = \frac{2}{3}$

$a = \frac{4}{r^2} \rightarrow a = \frac{4}{\left(\frac{2}{3}\right)^2} = \frac{4}{\left(\frac{4}{9}\right)} = 4 \times \frac{9}{4} = 9$

$\therefore\; n$th term: $T_n = 9\left(\frac{2}{3}\right)^{n-1}$

2. $T_n = 3r^{n-1} \rightarrow T_3 = 3r^2 = 48$ $\qquad \therefore\; r^2 = 16 \;\therefore\; r = \pm 4$, but $r > 0$, so $r = 4$

3. $5; x; y$ – arithmetic; $x; y; 81$ – geometric

$\therefore\ y - x = x - 5$ and $\frac{81}{y} = \frac{y}{x}$

$\therefore\ y = 2x - 5$ and $81x = y^2$ now we solve simultaneously…

$\therefore 81x = (2x - 5)^2$

$\therefore 81x = 4x^2 - 20x + 25$

$\therefore 4x^2 - 101x + 25 = 0$

$\therefore (4x - 1)(x - 25) = 0$ $\therefore\ x = 25$ (whole numbers only)

$\therefore y = 2(25) - 5 = 45$

4. A decrease of 10% is equivalent to finding 90% of the original amount.

$\therefore$ sequence: 2000 ; 2000 x (0,9) ; [2000 x (0,9)] x 0,9 ; …

2000 ; 2000 x (0,9) ; 2000 x $(0,9)^2$; 2000 x $(0,9)^3$

i.e. geometric sequence: $a = 2000$; $r = 0{,}9$

$\therefore T_{12} = 2000 \times (0{,}9)^{11}$

$= R627{,}62$ (December salary)

$\therefore$ no, you will earn less than R1 000

Activity 4

1. $T_1 = a = \frac{2}{3}$ and $T_5 = ar^4 = \frac{27}{8}$. $\therefore\ \frac{2}{3} \times r^4 = \frac{27}{8}\ \therefore\ r = \sqrt[4]{\frac{81}{16}} = \pm\frac{3}{2}$ but $r > 0$

$S_n = \frac{a(r^n - 1)}{r - 1}\ \therefore\ S_5 = \frac{\frac{2}{3}\left[\left(\frac{3}{2}\right)^5 - 1\right]}{\frac{3}{2} - 1} = 8{,}79$

2. Given the geometric series: $6 + 3 + \frac{3}{2} + \frac{3}{4} + \ldots \rightarrow a = 6; r = \frac{1}{2}$

2.1 $S_{11} = \frac{6\left[\left(\frac{1}{2}\right)^{11} - 1\right]}{\frac{1}{2} - 1} = 11{,}994$

2.2 $S_\infty = \frac{a}{1 - r} = \frac{6}{1 - \frac{1}{2}} = 12$

3. Given: $\sum_{n=1}^{\infty} 2r^{n-1} = 12$: $\left.\begin{array}{l} T_1 = 2 \\ T_2 = 2r \\ T_3 = 2r^2 \end{array}\right\}$

$S_\infty = \frac{a}{1 - r} = 12$

$\therefore\ \frac{2}{1 - r} = 12$ $\therefore\ 2 = 12 - 12r$

$12r = 10$

$\therefore\ r = \frac{5}{6}$

4. Find the total earnings: $a = 3\ 500$; $r = 1{,}05$; $n = 12$

Series: $3500 + 3500(1{,}05) + 3500(1{,}05)^2 + \ldots$

$S_{12} = \frac{3500[(1{,}05)^{12} - 1]}{1{,}05 - 1}$

$= R55\ 709{,}94$

$\therefore$ Tax rate 20% i.e. tax $= \frac{20}{100} \times 55\ 709{,}94 = R11\ 141{,}99$

5. Area Circle 1: $\pi \times (10)^2 = 100\pi$

Area Circle 2: $\pi \times (10 \times \frac{4}{5})^2 = \pi \times (10)^2 \times (\frac{4}{5})^2$

Area Circle 3: $\pi \times (10 \times \frac{4}{5} \times \frac{4}{5})^2 = \pi \times \left[10 \times (\frac{4}{5})^2\right]^2 = \pi (10)^2 \times (\frac{4}{5})^4$

$\therefore$ Series: $\pi \times (10)^2 + \pi (10)^2 \times (\frac{4}{5})^2 + \pi (10)^2 \times (\frac{4}{5})^4 + \ldots$

$\therefore a = 100\pi$ and $r = (\frac{4}{5})^2 = \frac{16}{25}$

$$\therefore S_\infty = \frac{9}{1-r} = \frac{100\pi}{1-\frac{16}{25}}$$

$$= \frac{100\pi}{\frac{9}{25}}$$

$$= 100\pi \times \frac{25}{9}$$

$$= \frac{2500\pi}{9} \text{ cm}^2$$

Activity 5

1. $a = 1$; $T_9 = a + 8d = 15$; substituting we get $d = \frac{7}{4}$

Middle term $= T_5 = a + 4d = 1 + 4 \times \left(\frac{7}{4}\right) = 8$

or $T_5 = \frac{T_1 + T_9}{2} = \frac{1+15}{2} = 8$ (alternate method)

2. $T_2 = ar = \frac{16}{3}$ and $T_7 = ar^6 = \frac{81}{2}$. Solving simultaneously $a = \frac{32}{9}$; $r = \frac{3}{2}$

$$S_5 = \frac{\frac{32}{9}\left[\left(\frac{3}{2}\right)^5 - 1\right]}{\frac{3}{2} - 1} = \frac{422}{9} = 46\frac{8}{9}$$

3. Series representing the cumulative growth: $10 + 5 + \frac{5}{2} + \ldots$

This is an infinite geometric series with $a = 10$ and $r = \frac{1}{2}$

$$S_\infty = \frac{a}{1-r} = \frac{10}{1-\frac{1}{2}} = 20$$

So the max growth is 20 cm, added to the original height 110 cm $\rightarrow$ max 130 cm

4. $T_4 + T_9 = 0 \therefore (a + 3d) + (a + 8d) = 0 \therefore 2a = -11d$

$S_6 = \frac{6}{2}(2a + (6-1)d) = 36$

$\therefore 2a + 5d = 12$ (solving simultaneously)

$\therefore -11d + 5d = 12 \therefore d = -2$ and $a = 11$

$\therefore T_5 = a + 4d = 11$cncn $+ 4(-2) = 3$

5. $S_n = 4n^2 + 1$ Note: $S_1 = 4(1)^2 + 1 = 5 = T_1$

Now $S_7 + T_8 = S_8$ so $T_8 = S_8 - S_7$

$T_8 = (4(8)^2 - 1) - (4(7)^2 - 1)$

$T_8 = (255) - (195) = 60$

6. Calculate: $\sum_{k=2}^{\infty} 8\left(\frac{1}{2}\right)^{k+2}$ $\quad a = 8 \times \left(\frac{1}{2}\right)^4 = \frac{1}{2}$; $T_2 = 8 \times \left(\frac{1}{2}\right)^5 = \frac{1}{4} \therefore r = \frac{1}{2}$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

7. Given: $\frac{1}{3}$; x; $\frac{16}{3}$

7.1 arithmetic $\rightarrow d = T_2 - T_1 = T_3 - T_2$

$\frac{16}{3} - x = x - \frac{1}{3} \therefore 2x = \frac{17}{3} \therefore x = \frac{17}{6}$

7.2 geometric $\rightarrow r = \dfrac{T_2}{T_1} = \dfrac{T_3}{T_2} \rightarrow \dfrac{\frac{16}{3}}{x} = \dfrac{x}{\frac{1}{3}} \rightarrow \dfrac{16}{3} \times \dfrac{1}{3} = x^2$

$\therefore x^2 = \dfrac{16}{9} \quad \therefore x = \pm\dfrac{4}{3}$

8. Given sequence: $8(x-2)^2;\ 4(x-2)^3;\ 2(x-2)^4;\ x \neq 2$

8.1 $r = \dfrac{T_2}{T_1} = \dfrac{4(x-2)^3}{8(x-2)^2} = \dfrac{x-2}{2}$

To converge: $-1 < \dfrac{x-2}{2} < 1 \rightarrow\ -2 < x-2 < 2\ \therefore\ 0 < x < 4$

8.2 S_∞ if $x = 2{,}5$: $8(2{,}5-2)^2;\ 4(2{,}5-2)^3;\ 2(2{,}5-2)^4$

$8\left(\frac{1}{2}\right)^2;\ 4\left(\frac{1}{2}\right)^3;\ 2\left(\frac{1}{2}\right)^4\ \therefore\ a = 2;\ r = \dfrac{1}{4}$

$S_\infty = \dfrac{a}{1-r} = \dfrac{2}{1-\frac{1}{4}} = \dfrac{8}{3}$

9. Given: $\sum_{p=1}^{n}(2p-3) = 120 \qquad \left.\begin{aligned} T_1 &= 2(1)-3 = -1 \\ T_2 &= 2(2)-3 = 1 \\ T_3 &= 2(3)-3 = 3 \end{aligned}\right\}\ a = -1;\ d = 2$

$S_n = \frac{n}{2}[2a + (n-1)d] = 120\ \ \therefore\ \frac{n}{2}[-2 + (n-1)2] = 120\ \ \therefore\ n(2n-4) = 240$

$\therefore 2n^2 - 4n - 240 = 0\ (\div 2)$

$\therefore n^2 - 2n - 120 = 0 \rightarrow (n-12)(n+10) = 0\ \ \therefore\ n = 12$ (reject $n = -10$)

Activity 6

1. $\quad T_n = -5 + (n-1)(4)$

$439 = -5 + 4(n-1)$

$444 = 4(n-1)$

$n - 1 = 111$

$n = 112$

2.1 14 (Note: This question is easier if you recognise that two sequences have been combined).

2.2 The sum of 1's is given by $S_{25} = (1 \times 25) = 25$

The sum of the other numbers in the sequence is given by

$S_{25} = \frac{n}{2}[2a + (n-1)d]$

$= \frac{25}{2}[2\cdot2 + (25-1)3]$

$= 950$

$\therefore$ The required sum is $25 + 950 = 975$

3.1 $r = \dfrac{27p^2}{81p} = \dfrac{p}{3}$

3.2 $-1 < \frac{p}{3} < 1\ \ \therefore\ -3 < p < 3; \quad p \neq 0$

3.3 If $p = 2$ the sequence is $162\ ;\ 108\ ;\ 72\ ;\ 48\ ;\ \ldots\ldots\ \therefore\ a = 162;\ r = \frac{2}{3}$

$S_\infty = \dfrac{a}{1-r}\ \ \therefore\ S_\infty = \dfrac{162}{1-\frac{2}{3}} = 486$

4.1 Tebogo's sequence will form a geometric sequence with common ratio 3. Thembe's sequence will form a **quadratic sequence** with a constant second difference of 8.

4.2 Thembe: $\begin{array}{ccccccc} 2 & & 6 & & 18 & & 38 \\ & 4 & & 12 & & 20 & \\ & & 8 & & 8 & & \end{array}$

1st difference

2nd difference (constant difference = 8)

$\therefore$ quadratic sequence of the form: $T_n = an^2 + bn + c$

$\therefore 2a = 8 \;\therefore a = 4 \;\therefore T_n = 4n^2 + bn + c$

sub $n = 1$: $2 = 4 + b + c$

$\therefore -2 = b + c$

sub $n = 2$: $6 = 4(2)^2 + 2b + c$

$\therefore -10 = 2b + c$

$\therefore b = -8$ and $c = 6$

$\therefore T_n = 4n^2 - 8n + 6$

Please note that there are many different ways to obtain the general equation for this sequence. Quadratic, or second difference, sequences were covered in Grade 11.

Tebogo: Geometric sequence with common ratio, $r = 3 \;\therefore T_n = 2 \cdot 3^{n-1}$

4.3 from $T_n = 4n^2 - 8n + 6 \;\therefore T_{11} = 4(11)^2 - 8(11) + 6 \;\therefore T_{11} = 402$

4.4 $S_n = \dfrac{a(r^n - 1)}{r - 1} \;\therefore 531\,440 = \dfrac{2(3^n - 1)}{3 - 1} \;\therefore 531\,440 = 3^n - 1$

$\therefore 531\,440 = 3^n \;\therefore 3^{12} = 3^n \;\therefore n = 12$

5.1 $r = \dfrac{100}{50} = \dfrac{200}{100} = 2$; $a = \text{R}0{,}5$ It is better to work with Rands, $a = 0{,}5$ than cents, $a = 50$

$S_n = \dfrac{a(r^n - 1)}{r - 1} \;\therefore S_{10} = \dfrac{0{,}5(2^{10} - 1)}{2 - 1} = 511{,}5 \;\therefore$ total amount = R511,50

5.2 Yes; R511,50 + R290 = R801,50 $\therefore$ he will have enough money to buy the boots.

6.1 Pattern 3

$\frac{1}{4} + \frac{3}{16} + \frac{9}{64}$

Pattern 4

$\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256}$

6.2 $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \ldots \frac{3^{n-1}}{4^n} = \sum_{k=1}^{n} \frac{3^{k-1}}{4^k}$

6.3 Recognise that the word 'indefinitely' in the question implies → "sum to infinity".

$a = \frac{1}{4}; r = \frac{3}{4} \;\therefore S_\infty = \dfrac{a}{1 - r} = \dfrac{\frac{1}{4}}{1 - \frac{3}{4}} = 1$