# TRIGONOMETRY

*Compound & Double angle formulae*

LESSON 9

In order to master this section you must first learn the formulae, even though they will be given to you on the matric formula sheet.

We call these formulae the **compound angle formulae** since the 'angle' in question is not a single angle but the combination of two (or more) angles.

$$\left.\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta\end{aligned}\right\}$$

same sign

Notice that the **sin** expansion alternates between the sin and cos functions, and it 'keeps' the original sign between the terms.

$$\left.\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta\end{aligned}\right\}$$

change sign

Notice that the **cos** expansion repeats cos cos and then sin sin, and 'changes' the original sign between the terms.

So, how can we use these formulae to help solve problems?

Can you remember the **special angles**? Besides 0° and the multiples of 90°, we referred to 30°, 45° and 60° as special angles since we could evaluate the trig functions of these angles without the use of a calculator. Now, by using the new **compound angle formulae**, any sum or difference of two special angles results in a new special angle – which we can evaluate without the use of a calculator!

So, 15° is a 'special angle', since it is formed by 45° – 30° or 60° – 45°.

Similarly, 75° is also a 'special angle', formed by 30° + 45°, and so the list goes on.

So, to evaluate sin 15° we can rewrite it as sin (45° – 30°). Then we expand using the formulas above…

$$\begin{aligned}\sin(45°-30°) &= \sin 45°\cdot\cos 30° - \cos 45°\cdot\sin 30° \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \text{ (at this stage use a calculator)} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Try to see for yourself that sin 15°could have been expanded as sin (60° – 45°) to give the same answer.

In a similar way we can evaluate cos 75°:

$$\begin{aligned}\cos 75° = \cos(30°+45°) &= \cos 30°\cdot\cos 45° - \sin 30°\cdot\sin 45° \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

You should notice that cos 75° gave exactly the same answer as sin 15°. This was no coincidence. You should remember a rule from grade 11… Both cos and sin are co-functions, and since their angles are complementary (15° + 75° = 90°), they are equal.

This can be shown, using the appropriate reduction formula, as follows:

$$\sin\theta = \cos(90°-\theta)$$

$$\therefore \sin 15° = \cos(90°-15°) = \cos 75°$$

Did you know that we can now verify the co-function reduction formulae using our knowledge of compound angles? Let's see…

$\cos(90°-\theta) = \cos 90°\cdot\cos\theta + \sin 90°\cdot\sin\theta$ (Note: cos 90° = 0 and sin 90° = 1)

$= 0 \cdot \cos\theta + 1 \cdot \sin\theta$

$= \sin\theta$

Example

## Examples

Rewrite each of the following as the sin or cos of a single angle and evaluate where possible.

1. $\cos 40° \cdot \cos 50° - \sin 40° \cdot \sin 50°$
2. $\sin 110° \cdot \cos 10° - \cos 290° \cdot \cos 80°$

Solution

## Solutions

1. $\cos 40° \cdot \cos 50° - \sin 40° \cdot \sin 50°$

$= \cos(40° + 50°)$

$= \cos(90°)$

$= 0$

We notice that we have the cos·cos, sin·sin expansion, which means that we have the cos of a compound angle. Now we write the expansion as a **single cos of the repeated angles**. Remember to **change** the sign found between the terms since we are working with cos…

2. $\sin 110° \cdot \cos 10° - \cos 290° \cdot \cos 80°$

$= \sin(180° - 70°) \cdot \cos 10° - \cos(360° - 70°)\cos 80°$

$= \sin 70° \cdot \cos 10° - \cos 70° \cdot \cos 80°$

This example is significantly more complicated. Before we can even consider the 'type' of expansion it is advisable to always express all angles as equivalent acute angles, using the appropriate reduction formulae.

Now the expansion is starting to take on a more familiar form… but, it goes sin·cos, cos·cos. So more manipulation is required.

Remember from earlier that $\cos 80° = \sin 10°$

$= \sin 70° \cdot \cos 10° - \cos 70° \cdot \sin 10°$

$= \sin(70° - 10°) = \sin 60°$

$= \frac{\sqrt{3}}{2}$ special angle

We notice that we have the sin·cos, cos·sin expansion, which means that we have the sin of a compound angle. We write the expansion as a **single sin using the repeated angles**. Remember to **keep** the sign found between the terms since we are working with sin…

Activity

## *Activity 1*

1. Verify the following using the compound angle formulae:

1.1 $\cos(90° + \theta) = -\sin\theta$

1.2 $\sin(360° - \theta) = -\sin\theta$

2. Rewrite the following as the sin or cos of a single angle and evaluate where possible.

2.1 $\sin x \cdot \cos y - \sin y \cdot \cos x$

2.2 $\cos 2° \cdot \sin 88° + \sin 2° \cdot \cos 88°$



LIBERTY LIFE

2.3 $\cos 7x \cdot \cos 3x - \sin 7x \cdot \sin 3x$

2.4 $\cos 62° \cdot \cos 17° + \sin 118° \cdot \cos 73°$

Up until this point we have been content with combining angles which were different in nature. But what happens if, for example, we combine the same angle with itself?

We know that $\cos 60° = \frac{1}{2}$. But $\cos 60°$ can also be written as $\cos(30° + 30°)$ …

Applying the compound angle formulae:

$\boxed{\sin 30° = \frac{1}{2}}$ $\boxed{\cos 30° = \frac{\sqrt{3}}{2}}$

$$\begin{aligned}\cos(30° + 30°) &= \cos 30° \cdot \cos 30° - \sin 30° \cdot \sin 30° \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{3}{4} - \frac{1}{4} \\ &= \frac{2}{4} \\ &= \frac{1}{2}\end{aligned}$$

Confirming the result shown earlier.

We can generalise this finding, and others, by working with variables instead.

Lets see what happens when we expand $\sin 2\theta$ and $\cos 2\theta$.

Note: Both of these expansions are given on the matric formula sheet.

$\sin 2\theta = \sin(\theta + \theta) = \sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta$ (combining the like terms)

$= 2 \sin\theta \cdot \cos\theta$

i.e. $\boxed{\sin 2\theta = 2 \sin\theta \cdot \cos\theta}$

This means that $\sin 50°$ can be written as $2 \sin 25° \cdot \cos 25°$

Note: This double angle expansion is true for sin only. (We still need to explore what will happen when we expand the cos of a double angle).

e.g. Expanding $\sin 80°$ (If $2\theta = 80°$, then $\theta = 40°$)

$\sin 80° = 2 \underbrace{\sin 40°}_{(2 \sin 20° \cdot \cos 20°)} \cdot \cos 40°$

(But we can expand sin 40° again, similar to when we factorised differences of two perfect squares repeatedly in grade 9 and 10…)

$= 2(2 \sin 20° \cdot \cos 20°) \cdot \cos 40°$

$= 4 \sin 20° \cdot \cos 20° \cdot \cos 40°$

We can see that this process can continue indefinitely. However, you needn't worry about this. It will always be clear, in the questions that follow (as well as those in the final examination), when we have gone far enough… For now it is just important to be able to perform the expansion.

Now let's see what the cos expansion of a double angle looks like.

$\cos 2\theta = \cos(\theta + \theta) = \cos\theta \cdot \cos\theta + \sin\theta \cdot \sin\theta$

$= \cos^2\theta - \sin^2\theta$

i.e. $\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$

However, this result can be modified by using the special identity you learnt in grade 11:

from $\sin^2\theta + \cos^2\theta = 1 \rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

It is then possible to express the cos expansion of a double angle in terms of sin or cos only!

**For cos only**:

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\quad = \cos^2\theta - (1 - \cos^2\theta)$ $\quad$ $\boxed{\sin^2\theta = 1 - \cos^2\theta}$

$= \cos^2\theta - 1 + \cos^2\theta$

$= 2\cos^2\theta - 1$

$\therefore$ $\boxed{\cos 2\theta = 2\cos^2\theta - 1}$

**For sin only**:

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\quad = (1 - \sin^2\theta) - \sin^2\theta$ $\quad$ $\boxed{\cos^2\theta = 1 - \sin^2\theta}$

$= 1 - \sin^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\therefore$ $\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$

It is for this reason that the cos expansion of a double angle is listed as follows on the formula sheet:

$$\boxed{\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases}}$$

It will only become clearer to you, at a later stage, which form of the expansion is more useful and why, based on the specifics of each individual question.

e.g. Expanding cos 80° $\quad$ (If $2\theta = 80°$, then $\theta = 40°$)

$\cos 80° = \cos^2 40° - \sin^2 40°$

or $= 2\cos^2 40° - 1$

or $= 1 - 2\sin^2 40°$

Here too the process of 'halving' the angle can continue indefinitely. Just notice that you will always have a choice as to how you 'expand' the cos of a ***double*** angle, to form the resultant expansion in terms of a ***single*** angle. This expansion can be done in terms of sin only, cos only or a combination of both functions.

Let's now focus on a number of examples where the use of double and compound angle formulae can be used to simplify or evaluate expressions. Later we will use this knowledge to prove identities and even solve equations.

But, always remember that a thorough knowledge of algebra is essential in order to simplify the trigonometric expressions that you will encounter…

e.g. $\quad$ Evaluate without the use of a calculator: $(\cos 15° + \sin 15°)^2$

$(\cos 15° + \sin 15°)^2 = \cos^2 15° + 2\sin 15°\cdot\cos 15° + \sin^2 15°$ (FOIL out the bracket)

$= (\cos^2 15° + \sin^2 15°) + (2\sin 15°\cdot\cos 15°)$ (Rearranged)

$\sin^2\theta + \cos^2\theta = 1$ $\qquad$ since $2\sin\theta\cdot\cos\theta = \sin 2\theta$

$\therefore \sin^2 15° + \cos^2 15° = 1$ $\qquad$ $\therefore 2\sin 15°\cdot\cos 15° = \sin(2 \times 15°) = \sin 30°$

$= 1 + \sin 30° = 1 + \left(\frac{1}{2}\right) = 1{,}5$

e.g. $\quad$ Express the following in terms of trig ratios of angle $x$:

(This is an important concept which will allow you to prove identities and, more importantly, to solve equations later on in this chapter.)

1. $\sin 3x$
2. $\cos 3x$

Now we can apply the double angle formulae learnt earlier since two of the angles are not 'single' yet.

1. $\sin 3x = \sin(2x + x) = \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$

Express as the sum of two angles so that we can use our compound angle formulae, and in doing so, decompose the angles into smaller 'pieces'…

$= (2 \sin x \cdot \cos x) \cdot \cos x + (\cos^2 x - \sin^2 x) \cdot \sin x$

$= 2 \sin x \cdot \cos^2 x + \sin x \cdot \cos^2 x - \sin^3 x$

$= 3 \sin x \cdot \cos^2 x - \sin^3 x$

Remember that we can expand cos $2x$ in three different ways. All 3 would have been acceptable here.

Apply the double angle formulae where necessary. Remember that we can expand cos $2x$ in three ways…

2. $\cos 3x = \cos (2x + x) = \cos 2x \cdot \cos x - \sin 2x \cdot \sin x$

$= (\cos^2 x - \sin^2 x) \cdot \cos x - (2 \sin x \cdot \cos x) \cdot \sin x$

$= \cos^3 x - \sin^2 x \cdot \cos x - 2 \sin^2 x \cdot \cos x$

$= \cos^3 x - 3 \sin^2 x \cdot \cos x$

## *Activity 2*

Activity

1. Evaluate the following without the use of a calculator:

   1.1 $2 \sin 15° \cdot \cos 15°$

   ............................................................

   1.2 $\cos^2 15° - \sin^2 15°$

   ............................................................

   1.3 $1 - 2 \sin^2 15°$

   ............................................................

   1.4 $2 \cos^2 22{,}5° - 1$

   ............................................................

   1.5 $\sin 22{,}5° \cdot \cos 22{,}5°$

   ............................................................

   1.6 $\cos 15°$

   ............................................................

2. Using the double angle formulae, derive a formula for $\sin x$ and $\cos x$ in terms of sin and cos of $\left(\frac{x}{2}\right)$.

   ............................................................

   ............................................................

3. Show that $\sin (45° + x) \cdot \sin (45° - x) = \frac{1}{2} \cos 2x$

   ............................................................

   ............................................................

As was mentioned at the beginning of this chapter, double and compound angles can be found in almost every aspect of trigonometry and can almost certainly be found, often in disguised form, in every trig question in the examination.

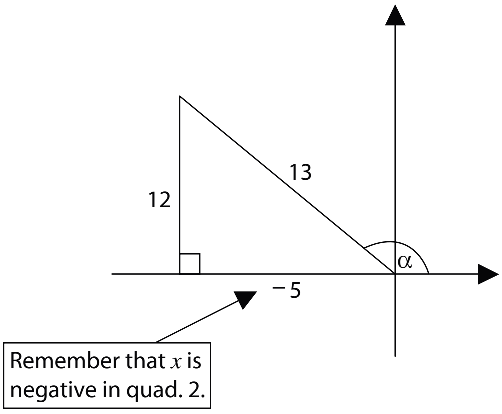
In the next few examples we will try to show you exactly what to look out for…

Let's start with typical examination type questions.

*Using diagrams to solve problems*

We have already seen the following type of question, in grade 11, where we are given a trig ratio and an additional piece of information (in order to isolate the specific quadrant). We then answer the questions that follow.

**Example 1:** If $\sin \alpha = \frac{12}{13}$ and $\tan \alpha < 0$, deduce $\cos 2\alpha + \tan \alpha$




The two pieces of information enable us to deduce that $\alpha$ is an angle in the second quadrant, where sin is positive AND tan is negative. By drawing the appropriate scale drawing, and using our knowledge of Pythagoras' theorem, we can also find the length of the missing side.

Now every question that follows hinges on the given diagram – where the given angle is $\alpha$. If we have a question that refers to angles of $2\alpha$ (or $3\alpha$ etc.) then we must use our knowledge of compound/double angles to **modify the question to refer back to our diagram.**

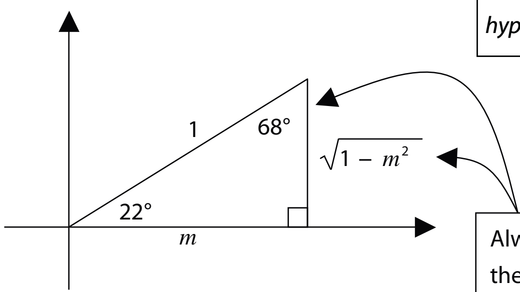
$$\cos 2\alpha + \tan \alpha = (2\cos^2 \alpha - 1) + \tan \alpha$$
$$= 2\left(\frac{-5}{13}\right)^2 - 1 + \left(\frac{12}{-5}\right)$$
$$= 2\left(\frac{25}{169}\right) - 1 + \left(\frac{12}{-5}\right)$$
$$= -\frac{2\,623}{845} \qquad \text{(use a calculator here)}$$

Now we can just substitute ratios from our diagram, since all the angles in ***our*** question allow us to refer back to the diagram which has $\alpha$ as our angle.

**Example 2:** If $\cos 22° = m$, express the following in terms of m:

1. sin 44°
2. cos 46°

Start by writing $\cos 22° = m$ as $\cos 22° = \frac{m}{1}$ when the ratio is not given as a fraction. Then we can assign $m$ to *adjacent* and 1 to *hypotenuse*, since $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.




Always find the remaining unknowns in the triangle. Here we find the **opposite** side using Pythagoras and the 3rd angle in the triangle. Both of these unknowns are of critical importance.

Once again the solution to this particular question hinges on the use of the sketch and the ability to rewrite the angles in the question in terms of the angles in the diagram.

Since the angles are 22° and 68° and we should always check to see if the angles in our question have been formed from the sum or difference of these angles (compound angle formulae) or by doubling (double angle formulae) the angles.

1. $\sin 44° = 2 \sin 22° \cdot \cos 22°$
$$= 2\left(\frac{\sqrt{1 - m^2}}{1}\right).m$$
$$= 2m.\sqrt{1 - m^2}$$

Which means we can now read easily from the sketch.

2. $\cos 46° = \cos (68° - 22°)$

$$= \cos 68° \cdot \cos 22° + \sin 68° \cdot \sin 22°$$
$$= \left(\frac{\sqrt{1-m^2}}{1}\right).(m) + (m).\left(\frac{\sqrt{1-m^2}}{1}\right)$$
$$= 2m.\sqrt{1-m^2}$$

Now use the (cos) compound angle expansion.

Note: It may have been easier to recognise cos46° = sin44° and then to use our answer from part 1.

### *Working with identities*

When working with identities we will usually be required to prove one side equal to the other. In the process we work with the left hand side (LHS) independently to the right hand side (RHS). As a general rule, as strange as it may seem, we work with the harder looking of the two sides.

Remember that almost always good algebra underpins the manipulation taking place.

Here are some basic steps to follow:

1. Always try to convert all terms to sin's and cos's. $\rightarrow \tan x = \frac{\sin x}{\cos x}$
2. When adding fractions, form LCD's and simplify. In doing so, through cross multiplying, the numerators can become modified… and often form the square identity $\sin^2 x + \cos^2 x = 1$. This identity can also be expressed as:

   $\sin^2 x = 1 - \cos^2 x$

   $\cos^2 x = 1 - \sin^2 x$
3. Look to see if we can use the compound or double angle formulae. It is usually easier to 'change' compound angles to single angles. So when one side of an identity contains single angles only and the other side a combination of compound/double and single angles this is the side that we must choose to work with.

Let's try to put this theory into practice…

e.g. Prove the identity: $\frac{\sin 2x + \cos 2x + 1}{\sin x + \cos x} = 2 \cos x$

Remember that we could choose any one of three expansions for cos $2x$. This expansion works best here.

LHS: $\frac{\sin 2x + \cos 2x + 1}{\sin x + \cos x} = \frac{(2 \sin x \cdot \cos x) + (2 \cos^2 x - 1) + 1}{\sin x + \cos x}$

We express each trig function in terms of $x$ only. The right hand side of the identity directs our thinking …

$$= \frac{2 \sin x \cdot \cos x + 2 \cos^2 x}{\sin x + \cos x} = \frac{2 \cos x.(\sin x + \cos x)}{\sin x + \cos x}$$
$$= \frac{2 \cos x.(\cancel{\sin x + \cos x})}{(\cancel{\sin x + \cos x})}$$
$$= 2 \cos x = \text{RHS}$$

Remember to factorise…

We can only consider cancelling out terms once the top and the bottom are expressed as products.

Let's consider another identity…

e.g. Prove: $\frac{2 \sin x \cdot \cos x}{\cos^4 x - \sin^4 x} = \tan 2x$

Although earlier it was suggested that we should always work with the side containing the double angles, and then express these as single angles, in this instance it is better to work with the LHS first.. There is 'more to do'

RHS $= \tan 2x = \frac{\sin 2x}{\cos 2x}$

It is possible to simplify both sides of an identity, so long as you work with the sides independently, as we have done here.

LHS: $\dfrac{2\sin x\cdot\cos x}{\cos^4 x - \sin^4 x} = \dfrac{\sin 2x}{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}$

factorise

"double angle"

special identity

$= \dfrac{\sin 2x}{(\cos 2x)(1)}$

$= \dfrac{\sin 2x}{\cos 2x} = \text{RHS}$

Factorise, since we have a difference of two perfect squares AND we have a single angle, so there is no need to use the double angle formulae.

Before attempting some identities, in the form of an exercise, let's try one more!

e.g. Prove: $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$

NOTICE THE FRACTIONS…

LHS: $= \dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = \dfrac{\sin 3x\cdot\cos x}{\sin x\cdot\cos x} - \dfrac{\cos 3x\cdot\sin x}{\sin x\cdot\cos x}$

LCD: $\sin x\cdot\cos x$

$= \dfrac{\sin 3x\cdot\cos x - \cos 3x\cdot\sin x}{\sin x\cdot\cos x}$

$= \dfrac{\sin(3x - x)}{\sin x\cdot\cos x} = \dfrac{\sin 2x}{\sin x\cdot\cos x}$

$= \dfrac{2\sin x\cdot\cos x}{\sin x\cdot\cos x}$

$= 2 = \text{RHS}$

Form the LCD so that we are able to *combine* the fractions. Once we have combined the numerators over a single denominator we should notice a very specific relationship… We have the compound angle expansion for sin, so we can simply reverse the process to obtain sin $2x$.

Further algebraic manipulation gives us the result we were looking for.

Activity

## *Activity 3*

1. Prove the following:

   1.1 $\tan x + \dfrac{\cos x}{\sin x} = \dfrac{2}{\sin 2x}$

   1.2 $2\sin^2(45° - x) = 1 - \sin 2x$

   1.3 $\dfrac{1 + \sin 2x}{\sin^2 x} = \left(1 + \dfrac{\cos x}{\sin x}\right)^2$

2. Prove: $\dfrac{\cos x - \cos 2x + 2}{3\sin x - \sin 2x} = \dfrac{1 + \cos x}{\sin x}$

3. Prove: $\tan x = \dfrac{1 - \cos 2x - \sin x}{\sin 2x - \cos x}$

*Solving equations*

In grade 11 you would have had lots of practice solving trig equations – even quadratic equations which required factorising and resulted in two or more solutions. You would have also given solutions for specific intervals as well as the general solution.

All that we are going to do now is to build on these concepts by incorporating our knowledge of compound/double angles.

e.g. Give the general solution for $2\sin^2 x + 3\sin x - 2 = 0$

Typical Grade 11 problem.

Note: There is no need to use our knowledge of double angles here as each trig function is 'operating' on the same angle. So we can just factorise as normal and then solve…

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -2$ (no solution)

$$\left.\begin{array}{l}\text{quadrant 1: } x = 30° + k{\cdot}360° \\ \text{quadrant 2: } x = 150° + k{\cdot}360°\end{array}\right\} k \in \mathbb{Z}$$

**Hint:** If you battle to factorise these quadratics, make a simple substitution.

Say, let $y = \sin x$ and substitute. So the quadratic becomes:

$2y^2 + 3y - 2 = 0$

$(2y - 1)(y + 2) = 0$

$\therefore\ y = \frac{1}{2}$ or $y = -2$

So what happens when our trig functions are operating on different angles, like $2x$ and $x$?

e.g. Solve for $x$: $\cos 2x - 1 = -3\cos x$, where $0° \leq x \leq 360°$

Start by rearranging the equation so that all the terms are taken to the same side.

$\cos 2x + 3\cos x - 1 = 0$

$$\cos 2x = \begin{cases}\cos^2 x - \sin^2 x \\ 1 - 2\sin^2 x \\ 2\cos^2 x - 1\end{cases}$$

Now look at the terms carefully. The key lies with the middle term $\rightarrow 3\cos x$, although this may only become apparent why at a later stage. Remember $\cos 2x$ can be expanded in any one of three ways BUT only one of these 'expansions' will work here. Can you see why?

By using the expansion $\cos 2x = 2\cos^2 x - 1$ we are able to rewrite our original equation as a quadratic in terms of cos only, so we can factorise normally.

$(2\cos^2 x - 1) + 3\cos x - 1 = 0$

$2\cos^2 x + 3\cos x - 2 = 0$ Which is almost identical to the earlier trinomial.

$(2\cos x - 1)(\cos x + 2) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -2$ (no solution)

quadrant 1: $x = 60°$

quadrant 4: $x = 300°$

e.g. Solve for $x$: $\cos 2x - 3\sin x = -1$, where $0° \leq x \leq 360°$

Change to a single angle, BUT this time in terms of sin only, so that we can form a *nice* quadratic.

$\cos 2x - 3\sin x = -1$

$(1 - 2\sin^2 x) - 3\sin x = -1$

$-2\sin^2 x - 3\sin x + 2 = 0$

$2\sin^2 x + 3\sin x - 2 = 0$ …

Rearrange to get the quadratic in the standard form, so that you can factorise.

Which gives us a familiar looking quadratic… which we solved earlier.

Hopefully you are starting to recognise how quadratics can be disguised BUT the big hint to use the double/compound angle formulae is in the comparison between the angles of each term. The cos double angle is always a very useful and powerful expansion since it can be rewritten in terms of sin or cos only, of the single angle.

So, what would the sin of a double angle look like in an equation, and how do we go about solving?

e.g. Determine the general solution to the equation $2 \sin 2x - 3 \cos x = 0$.

Not the same angle.

Always start by making a comparison of the angles which the trig functions are operating on. They should be operating on the same variable (both $x$'s) but they might have different sizes, in that one is double the other. This is the clear indictor that we must **change the angle** and write it as the trig function of a **single angle**.

When we change the sin double angle it is often easier than working with cos since it can be only be expanded in one way.

Factorise and then solve for each 'root' formed.

$2 \sin 2x - 3 \cos x = 0$

$2(2 \sin x \cdot \cos x) - 3 \cos x = 0$

$4 \sin x \cdot \cos x - 3 \cos x = 0$

$\cos x(4 \sin x - 3) = 0$

$\cos x = 0$ or $\sin x = \frac{3}{4}$

$\left.\begin{array}{l}\text{quadrant 1: } x = 0° + k \cdot 360° \\ \text{quadrant 4: } x = 360° + k \cdot 360°\end{array}\right\} k \in \mathbb{Z}$

$\left.\begin{array}{l}\text{quadrant 1: } x = 48{,}6° + k \cdot 360° \\ \text{quadrant 2: } x = 131{,}4° + k \cdot 360°\end{array}\right\} k \in \mathbb{Z}$

Essentially this is the same solution and so it is not necessary to give both here.

Remember to include the $k \cdot 360°$ part since we are giving the general solution to the equation.

Let's attempt one more problem before working through an exercise.

Remember at the end to answer the question, which asks for specific solutions within the given interval…

e.g. Solve for $x$ if: $\sin 2x + \sin x = 6 \cos x + 3$; $-180° \leq x \leq 180°$

$$\sin 2x + \sin x = 6 \cos x + 3$$

$$2 \sin x \cdot \cos x + \sin x = 6 \cos x + 3$$

$$2 \sin x \cdot \cos x + \sin x - 6 \cos x - 3 = 0$$

$$\sin x(2 \cos x + 1) + 3(-2 \cos x - 1) = 0$$

$$\sin x(2 \cos x + 1) - 3(2 \cos x + 1) = 0$$

$$(2 \cos x + 1)(\sin x - 3) = 0$$

$\cos x = -\frac{1}{2}$ or $\sin x = 3$ (no solution)

Take all terms to the one side and then convert $\sin 2x$ to $2 \sin x \cos x$ so that all the angles are the same. Then look to factorise. Here you should see that we form a common bracket through grouping, ending with the more familiar looking product.

It is a good idea to always write your solution as a general solution, even if not asked to do so. Then you can add/ subtract multiples of 360° to the solutions, **rejecting those that fall outside the specific intervals**.

Reference/key angle: $x = 60°$

quadrant 2: $x = (180° - 60°) + k \cdot 360° = 120° + k \cdot 360°$

$\therefore x = \{\cancel{-240°}; 120°; \cancel{480°}\}$

$k \in \mathbb{Z}$

quadrant 3: $x = (180° + 60°) + k \cdot 360° = 240° + k \cdot 360°$

$\therefore x = \{\cancel{-480°}; -120°; \cancel{240°}\}$

These are the negative quadrants for cos.

$\therefore$ the solutions that satisfy the interval $-180° \leq x \leq 180°$ are $x = \{-120°; 120°\}$

## *Activity 4*

Activity

1. Calculate the value(s) of $x$, $x \in [-90°; 270°]$ if $\sin x = \cos 2x - 1$

2. Solve for $x \in [-90°; 90°]$ if $\cos x = -\cos 2x - \sin 2x + \sin x + 1$

## *Problem solving – 2d & 3d trig*

Real world trig problems cannot always be expressed in two dimensions. In fact, more often than not, we need our diagrams to represent information in 3 dimensions.

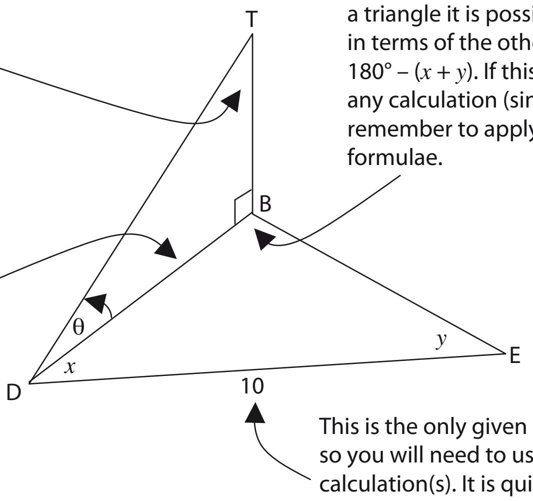
Trig can only ever be done in one triangle at a time. All 3-D and 2-D problems can be resolved into single triangles where information can be passed from one to the other. We must just remember to work with one triangle at a time and, where triangles/lines touch and share a common length, we are able to transfer information into another triangle.

We must also remember that knowledge of the sine and cosine rules is essential since we will often work with non right-angled triangles.

Let's look at an example to see what we mean. Probably before we start, let's analyse our diagram thoroughly, which is always a good starting point. (Try to see the relationship between the sides, the angles, and the 'boundary walls'.)

Since the sum of the angles in a triangle equals 180° this angle can be written as $90° - \theta$. So, if this angle is used in any calculation remember to apply the co-function reduction formulae: $\sin \leftrightarrow \cos$.

When there is one unknown angle in a triangle it is possible to express it in terms of the other two: here this is $180° - (x + y)$. If this angle is used in any calculation (sine or cosine rule) remember to apply your reduction formulae.



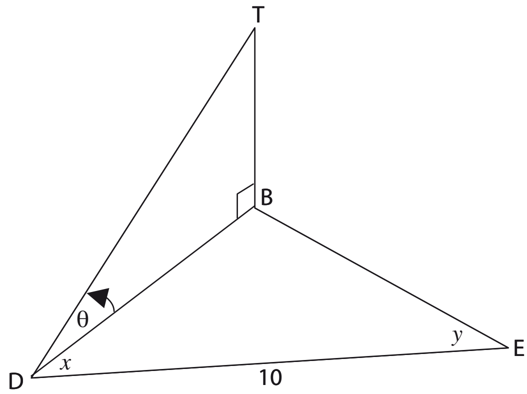

DB is a very important side since it connects the two different triangles. This will allow us to 'take' $\theta$ into triangle BED or $x$ and $y$ into triangle DBT. Always try to find the **common sides.**

This is the only given length… so you will need to use it in your calculation(s). It is quite possible that you may need to 'take' this value across into the triangle DBT.

Returning to our question…

In the diagram, B, D and E are points in the same horizontal plane. TB is a vertical telephone pole. The angle of elevation of T from D is $\theta$.

$B\hat{D}E = x$, $B\hat{E}D = y$ and DE = 10 m.




1. Express TB in terms of DB and $\theta$.

   Since we are working in a right-angled triangle we can use normal trigonometry. In $\triangle BDT$, $\tan \theta = \frac{TB}{DB}$

   $\therefore$ TB = DB.tan $\theta$

2. Show that $DB = \frac{10 \sin y}{\sin (x + y)}$

Look carefully at this question. The length 10 and the angles $x$ and $y$ are used, so we need to work within $\triangle BDE$. Since this is a non right-angles $\triangle$ we know that we must use the sine or cosine rule.

$$\frac{DB}{\sin y} = \frac{DE}{\sin D\hat{B}E}$$

From earlier we showed that $D\hat{B}E = 180° - (x + y)$

$$\frac{DB}{\sin y} = \frac{10}{\sin (180° - (x + y))}$$

Using the reduction formula: $\sin (180° - (x + y)) = \sin (x + y)$

We conclude…

$$DB = \frac{10 \sin y}{\sin (x + y)}$$

3. If $x = y$, show that $TB = \frac{5}{\cos y} \cdot \tan \theta$

The examiner will often set a question, divided into separate parts, like we have here. If you look carefully you should notice that part i) and part ii) are often 'linked' and direct our thinking in order to solve part iii). If you are stuck, always look to see if there is a link between the parts of a question…

From 1. TB = DB$\cdot$tan $\theta$, and

From 2. $DB = \frac{10 \sin y}{\sin (x + y)}$

so we conclude $TB = \frac{10 \sin y}{\sin (x+y)} \cdot \tan \theta$

At this stage we can use the additional given information: $x = y$. (Change all the $x$'s to $y$'s, since the final answer is expressed in terms of $y$.)

$$TB = \frac{10 \sin y}{\sin (y+y)} \cdot \tan \theta = \frac{10 \sin y}{\sin (2y)} \cdot \tan \theta$$

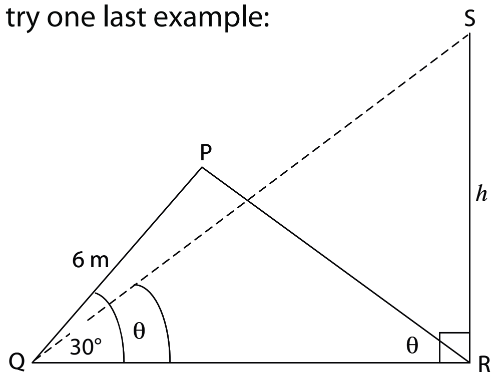
Remember our 'friend' the **double angle**?

$$TB = \frac{10 \cancel{\sin y}}{2 \cancel{\sin y} \cdot \cos y} \cdot \tan \theta = \frac{5}{\cos y} \cdot \tan \theta$$

In the previous example you will recall that although compound/double angle formulae were not required for each solution, this knowledge was still required in order to obtain the final solution. Let's try one last example:

In the diagram, P, Q and R are three points in the same horizontal plane and SR is a vertical tower of height h metres.

The angle of elevation of S from Q is $\theta$, $P\hat{R}Q = \theta$, $R\hat{Q}P = 30°$ and PQ = 6 m.




1. Express QR in terms of $h$ and a trig function of $\theta$.
2. Express $Q\hat{P}R$ in terms of $\theta$.
3. Hence, show that $h = 3(1 + \sqrt{3} \tan \theta)$

Solutions:

1. In $\triangle QRS$ $\tan \theta = \frac{h}{QR}$

   $\therefore QR = \frac{h}{\tan \theta}$

Start by locating $h$. Then try to find a triangle involving $\theta$ and $h$, which is quite easily done. (Especially since $\Delta QRS$ is right-angled.)

2. in $\triangle QRP$: $\hat{P} = 180° - (30° + \theta)$

   $\therefore \hat{P} = 150° - \theta$

Notice that although this triangle shares the same base (QR) as $\Delta QRS$ the triangles are in different *planes*.

3. Show: $h = 3(1 + \sqrt{3} \tan \theta)$

At this stage look to see if there is a link between part i), part ii) and part iii). Which there is… (and there usually will be).

From 1. $QR = \frac{h}{\tan \theta}$ and from 2. $\hat{P} = 150° - \theta$ so, using the sine rule in $\triangle PQR$:

Now expand, using the compound angle formulae for the sin expansion. Also note that you will encounter 'special angles'.

$$\frac{QR}{\sin \hat{P}} = \frac{PQ}{\sin (P\hat{R}Q)} \therefore \frac{QR}{\sin (150° - \theta)} = \frac{6}{\sin \theta} \therefore QR = \frac{6 \cdot \sin (150° - \theta)}{\sin \theta}$$

Now substitute the solution obtained in part i)… which is where the $h$ part comes into the answer: $QR = \frac{h}{\tan \theta}$

$$\frac{h}{\tan \theta} = \frac{6.(\sin 150° \cdot \cos \theta - \cos 150° \cdot \sin \theta)}{\sin \theta}$$

multiplied across

$$h = \frac{6.\left(\frac{1}{2} \cdot \cos \theta + \frac{\sqrt{3}}{2} \cdot \sin \theta\right)}{\sin \theta} \cdot \tan \theta$$

Multiply through by the 6 outside the bracket and then go about simplifying.

$$h = \frac{3 \cos \theta + 3\sqrt{3} \sin \theta}{\cancel{\sin \theta}} \cdot \frac{\cancel{\sin \theta}}{\cos \theta}$$

Remember to rewrite $\tan \theta$ as $\frac{\sin \theta}{\cos \theta}$

$$h = \frac{3\cos\theta}{\cos\theta} + \frac{3\sqrt{3}\sin\theta}{\cos\theta} = 3 + 3\sqrt{3}\tan\theta = 3(1 + \sqrt{3}\tan\theta)$$

Divide each term of the numerator by the resultant denominator (cos θ) and then simplify by factorising. Always remember to look at what is required in the question to direct your approach.

Activity

## *Activity 5: Mixed past paper questions*

1. If $\tan 53° = a$ and $\tan 31° = b$, show that $\sin 22° = \dfrac{a - b}{\sqrt{a^2+1}\cdot\sqrt{b^2+1}}$

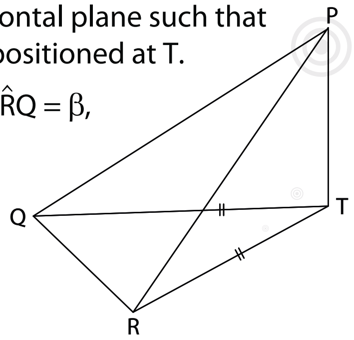
2.1 Prove: $\cos x \cdot \cos 2x + \dfrac{(\sin 2x)^2}{2\cos x} = \cos x$

2.2 Solve: $\cos A \cdot \cos 2A + \dfrac{(\sin 2A)^2}{2\cos A} = 0$ for $0° < A < 360°$

3. Determine x without using a calculator if

$\cos 70° \cdot \cos 10° + \sin 70° \cdot \sin 10° = 2\cos^2 2x - 1$ and $0° \leq 4x \leq 90°$

4. In the figure, Q, T and R are points in the horizontal plane such that TQ = TR = $y$ and TP represents a vertical pole positioned at T.




4.1 If the angle of elevation of P from Q is $\alpha$ and $P\hat{R}Q = \beta$,

Show that PQ = PR

4.2 Express PQ in terms of $y$ and a trig ratio of $\alpha$.

4.3 Prove that $QR = \frac{2y \cos \beta}{\cos \alpha}$

## Solutions to Activities

### Activity 1

1.1 $\cos (90° + \theta) = \cos 90° \cdot \cos \theta - \sin 90° \cdot \sin \theta = -\sin \theta$ ($\cos 90° = 0$; $\sin 90° = 1$)

1.2 $\sin (360° - \theta) = \sin 360° \cdot \cos \theta - \cos 360° \cdot \sin \theta$

$= -\sin \theta$ ($\sin 360° = 0$; $\cos 360° = 1$)

2.1 $\sin x \cdot \cos y - \sin y \cdot \cos x = \sin (x - y)$

2.2 $\cos 2° \cdot \sin 88° + \sin 2° \cdot \cos 88° = \sin (88° + 2°) = \sin 90° = 1$

2.3 $\cos 7x \cdot \cos 3x - \sin 7x \cdot \sin 3x = \cos (7x + 3x) = \cos 10x$

2.4 $\cos 62° \cdot \cos 17° + \sin 62° \cdot \sin 17° = \cos (62° - 17°) = \cos 45° = \frac{\sqrt{2}}{2}$

### Activity 2

1.1 $2 \sin 15° \cdot \cos 15° = \sin 30° = \frac{1}{2}$

1.2 $\cos^2 15° - \sin^2 15° = \cos (2 \times 15°) = \cos 30° = \frac{\sqrt{3}}{2}$

1.3 $1 - 2\sin^2 15° = \cos(2 \times 15°) = \cos 30° = \frac{\sqrt{3}}{2}$

1.4 $2\cos^2 22{,}5° - 1 = \cos(2 \times 22{,}5°) = \cos 45° = \frac{\sqrt{2}}{2}$

1.5 $\sin 22{,}5° \cdot \cos 22{,}5° = \frac{1}{2}(2\sin 22{,}5° \cdot \cos 22{,}5°) = \frac{1}{2}(\sin 45°) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$

1.6 $\cos 15° = \cos(45° - 30°) = \cos 45° \cdot \cos 30° + \sin 45° \cdot \sin 30° = \frac{\sqrt{6} + \sqrt{2}}{4}$

2. $\sin x = 2\sin\frac{x}{2} \cdot \cos\frac{x}{2}$

$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$

3. $\sin(45° + x) \cdot \sin(45° - x)$

$= [\sin 45° \cdot \cos x + \cos 45° \cdot \sin x][\sin 45° \cdot \cos x - \cos 45° \cdot \sin x]$

$= \left[\frac{\sqrt{2}}{2} \cdot \cos x + \frac{\sqrt{2}}{2} \cdot \sin x\right]\left[\frac{\sqrt{2}}{2} \cdot \cos x - \frac{\sqrt{2}}{2} \cdot \sin x\right]$ … diff of 2 perfect squares

$= \left(\frac{\sqrt{2}}{2}\right)^2 \cos^2 x - \left(\frac{\sqrt{2}}{2}\right)^2 \sin^2 x = \frac{1}{2}(\cos^2 x - \sin^2 x) = \frac{1}{2}\cos 2x$

**Activity 3**

1.1 RTP: $\tan x + \frac{\cos x}{\sin x} = \frac{2}{\sin 2x}$

LHS: $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos x} = \frac{1}{\sin x \cdot \cos x} = \frac{2}{2\sin x \cdot \cos x} = \frac{2}{\sin 2x}$

1.2 RTP: $2\sin^2(45° - x) = 1 - \sin 2x$

NOTE: from: $1 - 2\sin^2 a = \cos 2a \rightarrow 2\sin^2 a = 1 - \cos 2a$ (if $a = 45° - x$)

LHS: $2\sin^2(45° - x) = 1 - \cos 2[(45° - x)] = 1 - \cos(90° - 2x)$

$= 1 - \sin 2x = \text{RHS}$

1.3 RTP: $\frac{1 + \sin 2x}{\sin^2 x} = \left(1 + \frac{\cos x}{\sin x}\right)^2$

RHS: $\left(1 + \frac{\cos x}{\sin x}\right)^2 = \left(\frac{\sin x + \cos x}{\sin x}\right)^2 = \frac{\sin^2 x + 2\sin x \cdot \cos x + \cos^2 x}{\sin^2 x}$

$= \frac{(\sin^2 x + \cos^2 x) + 2\sin x \cdot \cos x}{\sin^2 x} = \frac{1 + \sin 2x}{\sin^2 x}$

2. RTP: $\frac{\cos x - \cos 2x + 2}{3\sin x - \sin 2x} = \frac{1 + \cos x}{\sin x}$

LHS: $\frac{\cos x - \cos 2x + 2}{3\sin x - \sin 2x} = \frac{\cos x - (2\cos^2 x - 1) + 2}{3\sin x - 2\sin x \cdot \cos x} = \frac{-(2\cos^2 x - \cos x - 3)}{\sin x(3 - 2\cos x)}$

$= \frac{-(2\cos x - 3)(\cos x + 1)}{\sin x(3 - 2\cos x)} = \frac{\cancel{(-2\cos x + 3)}(\cos x + 1)}{\sin x\cancel{(3 - 2\cos x)}}$

$= \frac{\cos x + 1}{\sin x} = \text{RHS}$

3. RTP: $\tan x = \frac{1 - \cos 2x - \sin x}{\sin 2x - \cos x}$

RHS: $\frac{1 - \cos 2x - \sin x}{\sin 2x - \cos x} = \frac{1 - (1 - 2\sin^2 x) - \sin x}{2\sin x \cdot \cos x - \cos x} = \frac{2\sin^2 x - \sin x}{\cos x(2\sin x - 1)}$

$= \frac{\sin x\cancel{(2\sin x - 1)}}{\cos x\cancel{(2\sin x - 1)}} = \tan x = \text{LHS}$

**Activity 4**

1. $\sin x = \cos 2x - 1,\ x \in [-90°; 270°]$

$\sin x = (1 - 2\sin^2 x) - 1 \rightarrow 2\sin^2 x + \sin x = 0$

$\sin x(2\sin x + 1) = 0$

$\sin x = 0$ OR $\sin x = -\frac{1}{2}$ (acute angle = 30°)

Q3: Q4:

$x = 90° + k \cdot 360°$ OR $x = 210° + k \cdot 360°;\ 330° + k \cdot 360°$ $\quad k \in \mathbb{Z}$

$x = \{-30°; 90°; 210°\}$

2. $\cos x = -\cos 2x - \sin 2x + \sin x + 1,\ x \in [-90°; 90°]$

$\cos x = -(1 - 2\sin^2 x) - (2\sin x \cdot \cos x) + \sin x + 1$

$\cos x = 2\sin^2 x - 2\sin x \cdot \cos x + \sin x \rightarrow$

$0 = 2\sin^2 x - 2\sin x \cdot \cos x + \sin x - \cos x$

$0 = 2\sin x(\sin x - \cos x) + 1(\sin x - \cos x)$

$(\sin x - \cos x)(2\sin x + 1) = 0$

Divide both sides by cos $x$. This is a very special 'trick'.

$\sin x = \cos x$

$\tan x = 1$

$x = 45° + k \cdot 180°$

$\sin x = -\frac{1}{2}$

$\left. x = 210° + k \cdot 360°;\ 330° + k \cdot 360° \right\} k \in \mathbb{Z}$

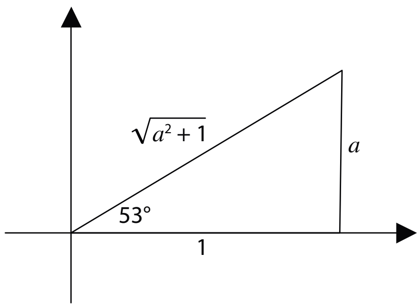
(from earlier)

$x = \{-30°; 45°\}$ for $x \in [-90°; 90°]$

**Activity 5**

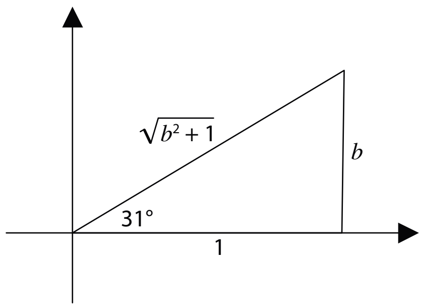
1. $\tan 53° = a$

$\tan 53° = \frac{a}{1}$




$\tan 31° = b$

$\tan 31° = \frac{b}{1}$




$\sin 22° = \sin(53° - 31°) = \sin 53° \cdot \cos 31° - \cos 53° \cdot \sin 31°$

$= \frac{a}{\sqrt{a^2+1}} \cdot \frac{1}{\sqrt{b^2+1}} - \frac{1}{\sqrt{a^2+1}} \cdot \frac{b}{\sqrt{b^2+1}} = \frac{a-b}{\sqrt{a^2+1} \cdot \sqrt{b^2+1}}$

2.1 RTP: $\cos x \cdot \cos 2x + \frac{(\sin 2x)^2}{2\cos x} = \cos x$

LHS: $\cos x \cdot \cos 2x + \frac{(2\sin x \cdot \cos x)^2}{2\cos x} = \cos x.(2\cos^2 x - 1) + \frac{4\sin^2 x \cdot \cos^2 x}{2\cos x}$

from: $(2\sin x\cos x)^2$

$= 2\cos^3 x - \cos x + 2\sin^2 x \cdot \cos x$

$= \cos x(2\cos^2 x - 1 + 2\sin^2 x)$

$= \cos x(2\cos^2 x + 2\sin^2 x - 1)$

$= \cos x(2 - 1) = \cos x$

2.2 Solve: $\cos A \cdot \cos 2A + \frac{(\sin 2A)^2}{2\cos A} = 0$ for $0° < A < 360°$

From 2.1: $\cos x \cdot \cos 2x + \frac{(\sin 2x)^2}{2\cos x} = \cos x$

$\therefore \cos A = 0$

$\therefore A = 90° + k \cdot 360°$ or $A = 270° + k \cdot 360°$

$\therefore A = 90°$ or $A = 270°$ (for $0° < A < 360°$)

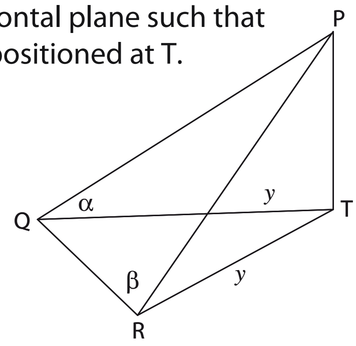
3. $\cos 70° \cdot \cos 10° + \sin 70° \cdot \sin 10° = 2\cos^2 2x - 1,\ 0° \leq 4x \leq 90°$

$\cos(70° - 10°) = \cos 4x \rightarrow \cos(60°) = \cos 4x$

$\therefore 4x = 60°$

$\therefore x = 15°$

4. In the figure, Q, T and R are points in the horizontal plane such that TQ = TR = $y$ and TP represents a vertical pole positioned at T.




4.1 Show that PQ = PR

Using Pythagoras, $PQ^2 = QT + PT^2$

$PR^2 = RT^2 + PT^2$

but QT = RT (= $y$, given)

$\therefore$ PQ = PR

4.2 Express PQ in terms of $y$ and a trig ratio of $\alpha$.

In $\triangle$PQT: $\cos \alpha = \frac{QT}{PQ}$

$\cos \alpha = \frac{y}{PQ}$

$\therefore PQ = \frac{y}{\cos \alpha}$

4.3 Prove that $QR = \frac{2y \cos \beta}{\cos \alpha}$

From 4.1: PQ = PR, so $P\hat{R}Q = \beta$ and $P\hat{Q}R = \beta$ (Isos $\triangle$)

$\therefore Q\hat{P}R = 180° - 2\beta$ (angles in a $\triangle$ = 180°)

Using sine rule in $\triangle$PQR: $\frac{QR}{\sin(180° - 2\beta)} = \frac{PQ}{\sin \beta}$

$$\frac{QR}{\sin(2\beta)} = \frac{PQ}{\sin \beta} \rightarrow QR = \frac{PQ \cdot \sin(2\beta)}{\sin \beta}$$

$$QR = \frac{PQ.2 \cancel{\sin \beta} \cdot \cos \beta}{\cancel{\sin \beta}}$$

$$= PQ.2 \cos \beta$$

$\boxed{\text{from: } \frac{y}{\cos \alpha}}$

$$QR = 4.2: \frac{y}{\cos \alpha} \cdot 2 \cos \beta$$

$$= \frac{2y \cos \beta}{\cos \alpha}$$

**Notes**