FINANCIAL MATHEMATICS

Simple Interest

As you learnt in grade 10, simple interest is calculated as a constant percentage of the money borrowed over a specific time period, for the complete period. So simple interest is based only on the amount of money invested / borrowed and not on a balanced basis.

Remember that:	Simple Depreciation: This is when a value is reduced at a rate of simple interest. The formula is constructed as follows:
If $F_V > P_V: F_V = P_V(1+in) \rightarrow$ appreciation If $F_V < P_V: F_V = P_V(1-in) \rightarrow$ depreciation	$F_V = P_V - I_1 - I_2 - I_3 - \ldots\ldots - I_n$ $\therefore\ F_V = P_V - \underbrace{i \cdot P_V}_{\text{1st year}} - \underbrace{i \cdot P_V}_{\text{2nd year}} - \underbrace{i \cdot P_V}_{\text{3rd year}} - \ldots - \underbrace{i \cdot P_V}_{\text{nth year}}$ $\therefore\ F_V = P_V - \underbrace{i \cdot P_V (1 + 1 + 1 + \ldots + \text{to } n \text{ terms})}_{\text{remove the common factor and add the n terms}}$ $\therefore\ F_V = P_V - i \cdot P_V\,(n)$ $\therefore\ F_V = P_V\,(1 - i \cdot n)$

F_V is the Future value of our money, P_V is the present value of our money, n is the term of the loan, and i is the interest rate given. A number of textbooks and the formula sheet you will get in your matric paper will show F_V as A and P_V as P.

In some cases we make investments which are costly due to the fact that our investment is not **appreciating** in value, but rather **depreciating.** If we buy a car, the moment it leaves the showroom, it loses value and is worth less than what we paid for it. We then say that it depreciated in value, and instead of adding interest, we subtract in our formulae.

Simple (or straight line) Depreciation

Example

Example 1

John bought a brand new Toyota at R250 000. In five years time he would like to replace this vehicle with a new one. John decides to work on a depreciation rate of 8% per annum on the straight line basis. What can the expected book value of this vehicle be five years from now?

Solution

Solution

$F_V = P_V(1 - i \cdot n)$
$\therefore F_V = 250\,000(1-(0{,}08) \cdot 5)$
$\therefore F_V = 250\,000(0{,}6)$
$\therefore F_V = \text{R}150\,000$

Example 2

R17 000 was invested for a period of 3 years and depreciated to an amount of R9 832. Determine the flat rate at which the money depreciated.

Solution

Since $F_V < P_V$, the investment depreciated. So:
$9\,832 = 17\,000(1 - i \cdot 3)$
$\therefore (1-3i) = \frac{9\,832}{17\,000}$
$\therefore -3i = \frac{9\,832}{17\,000} - 1$
$\therefore i = -\frac{1}{3}\left(\frac{9\,832}{17\,000} - 1\right) = 0{,}14054\ldots$
$\therefore \text{rate} = 100i = 14{,}05\%$ per annum

Compound Depreciation

What separates simple depreciation from compound depreciation is that **simple depreciation is based on the original amounts** only, whereas compound depreciation is based on the reducing balance basis.

Let us consider the same examples as we did for simple depreciation, but this time with the depreciation calculated on the reducing balance scale:

Example 1

John bought a brand new Toyota at R250 000. In five years time he would like to replace this vehicle with a new one. John decides to work on a depreciation rate of 8% per annum on the reducing balance. What will the value of this vehicle be five years from now?

Solution

$F = P(1 - i)^n$

$\therefore F = 250\ 000\ (1 - 0{,}08)^5$

$\therefore F_V = 250\ 000\ (0{,}6590815232)$

$\therefore F_V = \text{R } 164770,\ 38$

Example 2

R17 000 was invested for a period of 3 years and depreciated to an amount of R9 832. Determine the depreciation rate at which the investment depreciated on the reducing balance.

Solution

Since $F_V < P_V$, the investment depreciated. So

$F_V = P_V(1 - i)^n$

$\therefore 9\ 832 = 17\ 000\ (1 - i)^3$

$\therefore (1 - i)^3 = \frac{9\ 832}{17\ 000}$

$\therefore -i = \sqrt[3]{\frac{9\ 832}{17\ 000}} - 1 = -0{,}1668350667$

$\therefore i = 0{,}1668$

But rate $= 100 \times i$

$\therefore$ rate = 16,68% per annum

Activity 1

1. A second hand motor car costing R120 000 is expected to have a lifetime of at least another 8 years. Thereafter it will be sold and the money used as a deposit on a new vehicle. If the depreciation rate is 5% p.a. on a linear scale, how much will be available as a deposit on the new vehicle 8 years from now?

2. A small town in the Karoo has a population of 15 234. A drought is causing people to move closer to the cities. This results in a loss of 8% per annum in the population. How many people will be in this town in five years if measured on

2.1 Compound depreciation

2.2 Straight line depreciation

Nominal and Effective interest rates

When working with problems involving interest, we use the term **payment period** as follows:

- Annually Once a year
- Semi-annually Twice a year
- Quarterly 4 times a year
- Monthly 12 times a year

If the interest due at the end of a payment period is added to the principal, so that the interest computed for the next payment period is based on this new amount formed by the old principal plus interest, then the interest is said to have been **compounded.** C**ompound interest** is interest paid on the initial principal and previously earned interest.

In the financial world we come across two different "types" of rates, the **nominal rate** and the **effective rate.** In any calculation that we do, we have to work with the effective rate.

We have not yet established how to compare interest rates offered by two financial institutions. For example, if Bank A offers you a rate of 14% per annum compounded monthly and Bank B offers a rate of 15% per annum compounded semi annually, which offer should we accept? The only way to really make such a choice is to "translate" both these rates into rates that **read the same**, that is per annum compounded annually, or per annum compounded monthly, etc. This is the rate where the ***stated period* and the *compounding periods* are the same**. These rates we refer to as **effective rates**. We cannot compare them otherwise.

$$12\%\ \underbrace{\text{per annum}}_{\text{stated period}}\ \underbrace{\text{compounded monthly}}_{\text{compounding period}}$$

We will compare the following rates to establish the relationship between nominal end effective rates.

Nominal rate	**Effective rate per period**
12% per annum, compounded monthly. This is a nominal monthly rate since the per annum (stated period) and the compounded monthly (compounding period) differ.	Now if interest is calculated at 12% per annum, and the compounding takes place monthly, then we will compound interest 12 times a year, since a year has twelve months. Thus $r = \frac{12}{12}\%$ = 1% per month compounded monthly. This rate is referred to as an effective rate per period. Note that the **per annum** changed to **per month**.
12% p.a. compounded semi-annually.	$\frac{12}{2}\%$ = 6% per semi-annum compounded semi annually.
12% p.a. compounded quarterly.	$\frac{12}{4}\%$ = 3% per quarter compounded quarterly.

Example 1

Change a nominal rate of 14% p.a. compounded weekly to an equivalent effective monthly rate.

Solution

$$\left(1+\frac{i_{12}}{12}\right)^{12}=\left(1+\frac{i_{52}}{52}\right)^{52}$$

$$\therefore\ 1+\frac{i_{12}}{12}=\left(1+\frac{0,14}{52}\right)^{\frac{52}{12}}$$

$$\therefore\ \frac{i_{12}}{12}=1,011719127\ldots-1$$

$$\therefore\ \frac{i_{12}}{12}=0,011719127\ldots$$

rate $= 100 \times 0,0117\ldots$% p.m.c.m

So rate = 1,17% per month compounded monthly.

Example 2

Change a nominal monthly rate of 16% p.a. compounded monthly to an equivalent effective semi-annual rate.

Solution

$$\left(1+\frac{i_{2}}{2}\right)^{2}=\left(1+\frac{i_{12}}{12}\right)^{12}$$

$$\therefore\ 1+\frac{i_{2}}{2}=\left(1+\frac{0,16}{12}\right)^{6}$$

$$\therefore\ \frac{i_{2}}{2}=1,08271455\ldots-1$$

$$\therefore\ \frac{i_{2}}{2}=0,08271455\ldots$$

rate $= 100 \times 0,0827\ldots$ p.s.a c.s.a.

So rate = 8,27% per semi-annum compounded semi- annually.

So in general this information affects the compounding period in the following way:

Effective annually	Effective semi-annually	Where:
$F_v = P_v(1+i)^n$	$F_v = P_v\left(1+\frac{i_2}{2}\right)^{2n}$	$i = \frac{\text{rate}}{100}$ n = years P_v = present value F_v = Future value
Effective monthly	**Effective quarterly**	
$F_v = P_v\left(1+\frac{i_{12}}{12}\right)^{12n}$	$F_v = P_v\left(1+\frac{i_4}{4}\right)^{4n}$	

Example 1

Which investment will be the best, 13% simple interest for two years or 12% p.a. compounded monthly for two years?

Solution

At simple interest:

$F_V = P_V(1+0,13\times 2) = 1,26P_V$

At compound interest:

$F_V = P_V\left(1+\frac{0,12}{12}\right)^{24} = 1,27P_V$ (2 d.p.)

So the investment at compound interest is better.
Remember you should not automatically go for the higher rate, without considering the compounding periods.

Example 2

For any savings account, which is the better option: 7% p.a. compounded monthly or 7,5% p.a. compounded semi-annually?

Solution

Notice that we do not know the period of the investment. So we can only compare the two if they look the same.

$$F_V = P_V\left(1+\frac{0,07}{12}\right)^{12n} = P_V(1,07229...)^n \qquad F_V = P_V\left(1+\frac{0,075}{2}\right)^{2n} = P_V(1,0764...)^n$$

So from this we see that the rate of 7,5% p.a.compounded semi-annually is best.

Example

Example 3

R100 is invested for three years, at a rate of 14% p.a. compounded quarterly. Determine its future value.

Solution

Solution

$F_V = P_V(1 + i)^n$

$\therefore F_V = 100\left(1 + \frac{i_4}{4}\right)^{4n}$

$\therefore F_V = 100\left(1 + \frac{0,14}{4}\right)^{12}$

$\therefore F_V = \text{R}151,11$

Example

Example 4

How much must be invested now to realise R14 000 five years from now if the money is invested at:

4.1 12% p.a. compounded semi-annually

4.2 11% p.a. compounded quarterly

Solution

Solution

4.1 Semi-annually:

$$F_V = P_V\left(1 + \frac{i_2}{2}\right)^{2n}$$

$$14\,000 = P_V\left(1 + \frac{0,12}{2}\right)^{10}$$

$$\therefore P_V = \frac{14\,000}{\left(1 + \frac{0,12}{2}\right)^{10}}$$

$$\therefore P_V = 14\,000\left(1 + \frac{0,12}{2}\right)^{-10}$$

$$\therefore P_V = \text{R}7\,817,53$$

4.2 Quarterly:

$$F_V = P_V\left(1 + \frac{i_4}{4}\right)^{4n}$$

$$14\,000 = P_V\left(1 + \frac{0,11}{4}\right)^{20}$$

$$\therefore P_V = 14\,000\left(1 + \frac{0,11}{4}\right)^{-20}$$

$$\therefore P_V = \text{R}8\,137,51$$

Activity

Activity 2

1. How long will it take money to double if it is invested at a rate of

1.1 15% p.a. compounded monthly

....

....

....

....

....

....

1.2 12% p.a. compounded semi-annually

1.3 8% p.a. compounded daily

2. It takes 12 years for R4 500 to accumulate to R25 073. Find the effective annual rate.

3. R3 500 is invested at 14,4% p.a. compounded quarterly. After 6 months, R1 000 is added to the investment, and the amount is reinvested at 16% p.a. compounded monthly. Find the accumulated amount after five years.

Solutions to Activities

Activity 1

1. $F_V = P_V(1 - i \cdot n)$

$\therefore F_V = 120\,000[1 - (0{,}05)\cdot 8]$

$\therefore F_V = 120\,000(0{,}6)$

$\therefore F_V = \text{R}72\,000$

There will be R72 000 available.

2.1 $\therefore F_V = 15\,234(1 - 0{,}08)^5$

$\therefore F_V = 15\,234(0{,}92)^5$

$\therefore F_V = 10\,040{,}44792$

There will be approximately 10 040 people left in the town.

2.2 $\therefore F_V = 15\,234(1 - 0{,}08 \times 5)$

$\therefore F_V = 15\,234(0{,}6)$

$\therefore F_V = 9\,140{,}4$ i.e. approximately 9 140 people left.

Activity 2

1.1 $F_V = P_V\,(1+i)^n$

$\therefore 2 = 1\left(1 + \frac{0{,}15}{12}\right)^{12n}$

$\therefore 2 = (1{,}0125)^{12n}$

$\therefore 12n = \frac{\log 2}{\log 1{,}0125} = 55{,}79763\ldots$

$\therefore n = 4{,}649\ldots$ i.e. 4 years and 8 months

1.2 $F_V = P_V\,(1+i)^n$

$\therefore 2 = 1\left(1 + \frac{0{,}12}{2}\right)^{2n}$

$\therefore 2 = (1{,}06)^{2n}$

$\therefore 2n = \frac{\log 2}{\log 1{,}06} = 11{,}89566\ldots$

$\therefore n = 5{,}947\ldots$ i.e. 5 yrs and 11 mths

1.3 $F_V = P_V\,(1+i)^n$

$\therefore 2 = 1\left(1 + \frac{0{,}08}{365}\right)^{365n}$

$\therefore 2 = (1{,}000219178)^{365n}$

$\therefore 365n = \frac{\log 2}{\log 1{,}000219178}$

$\therefore 365n = 3162{,}831758$

$\therefore n = 8{,}665\ldots$ i.e. 8 years and 8 months

2. $25\,073 = 4\,500\left(1 + \frac{i}{12}\right)^{144}$

$\therefore \left(1 + \frac{i}{12}\right)^{144} = 5{,}571777778$

$\therefore \frac{i}{12} = \sqrt[144]{5{,}571777778} - 1$

$\therefore \frac{i}{12} = 0{,}011999\ldots$

$\therefore i = 0{,}14399\ldots$

$\therefore r = 14{,}4\%$ p.a. compounded annually.

3. We need an effective monthly rate from the quarterly rate:

$$\left(1 + \frac{i_{12}}{12}\right)^{12} = \left(1 + \frac{i_4}{4}\right)^4$$

$$\therefore \left(1 + \frac{i_{12}}{12}\right)^{12} = \left(1 + \frac{0{,}144}{4}\right)^4$$

$$\therefore \frac{i_{12}}{12} = \sqrt[3]{1 + \frac{0{,}144}{4}} - 1$$

$$\therefore \frac{i_{12}}{12} = 0{,}01185881266$$

So:

6 months = 2 quarters

Total of 5 years is 60 months minus first 6 months.

$$F_V = \left\{\left[3\,500\left(1 + \frac{0{,}144}{4}\right)^2\right] + 1\,000\right\}\left(1 + \frac{0{,}16}{12}\right)^{54}$$

compounded quarterly

compounded monthly

Now enter this all on one line on your calculator. Remember to include all the brackets.

$$= \text{R}9\,725{,}60$$