ALGEBRAIC EXPRESSIONS

In this section we will be looking at the following algebraic operations.

- Completion of the square.
- Addition and subtraction of fractions.
- Simplification of exponential expressions.
- Addition, subtraction, multiplication and division of surds.

Completion of the square

The method of completing the square is used to express a quadratic expression in the form $a(x - p)^2 + q$

Example 1

Consider the expression $x^2 + 2x - 24$

We will now follow the steps to complete the square.

Solution

Step 1 Make sure that the coefficient of x^2 (the number in front of x^2) is 1. In this case it is 1 already.

Step 2 Rewrite the expression in the form $x^2 + 2x \ldots -24$ (We just move the constant number away)

Step 3 Look at the number in front of x. In this example it is 2. Halve the number, square the number, add and subtract the answer.

$$x^2 + 2x + 1^2 - 1^2 - 24$$

Step 4 Now we can factorise the first 3 terms.

$$(x + 1)^2 - 25$$

Note that the sign in the bracket must be the same as the coefficient of x in the original expression.

The square has now been completed and our expression is now in the form $a(x - p)^2 + q$

Let's now have a look at an expression where the coefficient of x^2 is not equal to 1.

Example 2

$$2x^2 - 11x - 6$$

Solution

Step 1 Make the coefficient of $x^2 = 1$.

We will have to take out a common factor of 2 to achieve this. Please note that we may NOT divide by two as this is not an equation.

$$2\left(x^2 - \frac{11}{2}x - 3\right)$$

Step 2 Rewrite

$$2\left(x^2 - \frac{11}{2}x \ldots - 3\right)$$

Step 3 Halve, square, add and subtract

$$2\left(x^2 - \frac{11}{2}x + \left(\frac{11}{4}\right)^2 - \left(\frac{11}{4}\right)^2 - 3\right)$$

Step 4 Factorise the first 3 terms in the braket

$$2\left[\left(x - \frac{11}{4}\right)^2 - \frac{121}{16} - 3\right]$$

$$2\left[\left(x - \frac{11}{4}\right)^2 - \frac{169}{16}\right]$$

We multiply 2 (our common factor) into the bracket.

$$2\left(x - \frac{11}{4}\right)^2 - \frac{169}{8}$$

Although this may seem difficult at first practise is all that is needed. Practise the 4 steps by doing the following exercise.

Activity

Activity 1

Complete the square for each of the following. Write your answers in the form $a(x - p)^2 + q$

1. $x^2 - 12x + 34$

2. $a^2 - 8a + 15$

3. $-2b^2 - b - 4$

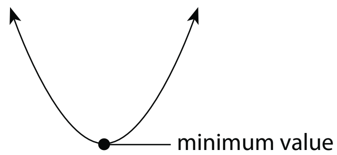
You might be asking yourself why is it necessary to complete the square?

Completion of the square will be used later on to solve quadratic equations and also to determine the turning points of parabolas. But more about that in later sections.

At this stage you need to be able to complete the square to determine the maximum or minimum value of a quadratic expression.

Consider the expression from Question 1 in activity 1
$x^2 - 12x + 34 = (x - 6)^2 - 2$.
The smallest value that this expression could ever have is -2.

You will learn later when dealing with functions that a parabola of the form $y = a(x - p)^2 + q$ where $a > 0$ has the following shape.



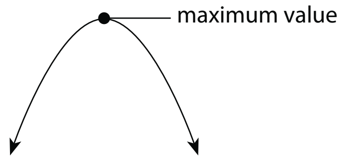
Consider the expression from Question 3 in activity 1

$-2b^2 - b - 4 =$

$-2\left(b + \frac{1}{4}\right)^2 - \frac{31}{8}$

The largest value that this expression will ever have is $\frac{-31}{8}$

This expression is said to have a maximum value. Expressions of the form $y = a(x - p)^2 + q$ where $a < 0$ are parabolas of the following shape:



Simplification of algebraic fractions

Multiplication and division

Consider the expression.

Example 1

$$\frac{3a^2 + 6a}{6a}$$

Solution

When simplifying a fraction such as this it is important to remember that we may NOT cancel over the + and – sign.

$$\frac{3a^2 + \cancel{6a}}{\cancel{6a}}$$ ✗ wrong

What we must do is factorise before we cancel.

$$\frac{3a(a + 2)}{6a}$$

Now we may cancel, and get

$$\frac{a + 2}{2}$$

Let's look at another example.

Example 2

$$\frac{2x^2 - 8}{3x^2 - 5x - 2}$$

Solution

In this expression we need to factorise both the numerator and the denominator.

$$\frac{2(x^2-4)}{(3x+1)(x-2)}$$

$$\frac{2(x+2)\cancel{(x-2)}}{(3x+1)\cancel{(x-2)}} = \frac{2(x+2)}{3x+1}$$

Example

Example 3

$$\frac{a^2-4}{a} \div \frac{4a-2a^2}{a}$$

Solution

Solution

We factorise the numerators and the denominators

$$\frac{(a-2)(a+2)}{a} \div \frac{2a(2-a)}{a}$$

The ÷ sign becomes a × sign when the second fraction is inverted.

$$\frac{(a-2)(a+2)}{a} \times \frac{a}{2a(2-a)}$$

Have a look at $(a-2)$ and $(2-a)$

Writing $(a-2)$ as $-(2-a)$, allows us to simplify further:

$$\frac{-\cancel{(2-a)}(a+2)}{\cancel{a}} \times \frac{\cancel{a}}{2a\cancel{(2-a)}}$$

$$= \frac{-(a+2)}{2a}$$

This is an important rule to remember:

$$b - a = -(a - b)$$

Now use the skills you have learnt to simplify the following:

Activity

Activity 2

Simplify:

1. $\dfrac{x^2-x-12}{12-3x}$

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2. $\dfrac{(x-2)(2x-4)-(x-2)(3x+4)}{(2-x)(x+8)}$

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3. $\dfrac{a^2}{16(a-1)} \times \dfrac{9a^4}{4} \div \dfrac{9}{8(a-1)^2}$

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Addition and subtraction of fractions

Consider the expression below:

Example 1

$$\frac{2}{x+5}+\frac{3}{x-3}$$

Solution

In order to simplify this fraction we need to get a common denominator.

$$\frac{2(x-3)+3(x+5)}{(x+5)(x-3)}$$

Once we have found the common denominator we multiply and simplify.

$$\frac{2x-6+3x+15}{(x+5)(x-3)}$$

We add the like terms.

$$\frac{5x+9}{(x+5)(x-3)}$$

Let's have a look at an example that has 3 fractions.

Example 2

$$\frac{2}{a-3}+\frac{5}{a+4}-\frac{7}{a+2}$$

Solution

We follow the same steps as before. First we find a common denominator.

$$\frac{2(a+4)(a+2)+5(a-3)(a+2)-7(a-3)(a+4)}{(a-3)(a+4)(a+2)}$$

Let's simplify by multiplying the terms in the numerator.

$$\frac{2(a^2+6a+8)+5(a^2-a-6)-7(a^2+a-12)}{(a-3)(a+4)(a+2)}$$

$$\frac{2a^2+12a+16+5a^2-5a-30-7a^2-7a+84}{(a-3)(a+4)(a+2)}$$

$$\frac{70}{(a-3)(a+4)(a+2)}$$

Now test your skills by doing the following examples:

Activity 3

Simplify:

1. $\frac{2}{x+3}-\frac{1}{(x+3)^2}$

2. $\frac{5}{3}+\frac{2}{x-2}+1\frac{1}{2}-\frac{3}{2(x-2)}$

3. $\dfrac{3}{x+2} - \dfrac{2}{3(x-1)} + \dfrac{1}{x-1}$

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Simplification of exponential expressions

Before we have a look at how to simplify exponential expressions let's revise the exponential laws.

1. $x^a \times x^b = x^{a+b}$

   e.g. $x^2 \times x^4 = x^6$

2. $x^a \div x^b = x^{a-b}$

   eg. $x^6 \div x^2 = x^4$

3. $(xy)^a = x^a y^b$

   eg. $(xy)^2 = x^2y^2$

4. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

   $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$

5. $(y^a)^b = y^{a \times b}$

   $(y^2)^3 = y^6$

Besides the laws of exponents the following is also important to remember.

- $x^0 = 1$
- $x^{-a} = \frac{1}{x^a}$
- $\frac{x^{-a}}{y^{-b}} = \frac{y^b}{a^a}$

We will now use our knowledge of exponential laws to simplify exponential expressions.

Let's start by looking at expressions involving multiplication and division only.

Example

Example 1

Simplify: $\dfrac{15^{x+1} \cdot 9^{3x-1} \cdot 25^x}{45^{2x} \cdot 27^{x-1} \cdot 5^x}$

Solution

Solution

Our first step is to change all the bases to prime numbers.

$$\frac{(5.3)^{x+1} \cdot (3^2)^{3x-1} \cdot (5^2)^x}{(3^2 \cdot 5)^{2x} \cdot (3^3)^{x-1} \cdot 5^x}$$

Now we use our knowledge of exponential laws to remove the brackets.

$$\frac{5^{x+1} \cdot 3^{x+1} \cdot 3^{6x-2} \cdot 5^{2x}}{3^{4x} \cdot 5^{2x} \cdot 3^{3x-3} \cdot 5^x}$$

All our bases are either 3 or 5. We will now multiply by adding the exponents and divide by subtracting.

$$5^{x+1+2x-2x-x} \cdot 3^{x+1+6x-2-4x-3x+3}$$

$$= 5^1 \cdot 3^2 = 45$$

Remember that the sign changes when we move an index from the denominator to the numerator.

Let's work through another example.

Example 2

Simplify: $\dfrac{9^{2n+1} \times 6^{2n-3}}{3^{5n-2} \times 2 \times 3^{n} \times 4^{n-2}}$

Solution

Again we write our bases as prime numbers.

$$\frac{(3^2)^{2n+1} \times (2 \times 3)^{2n-3}}{3^{5n-2} \times 2^1 \times 3^n \times (2^2)^{n-2}}$$

$$= \frac{3^{4n+2} \times 2^{2n-3} \times 3^{2n-3}}{3^{5n-2} \times 2^1 \times 3^n \times 2^{2n-4}}$$

$$= 3^{4n+2+2n-3-5n+2-n} \cdot 2^{2n-3-1-2n+4}$$

$$= 3^1 \cdot 2^0 = 3 \times 1 = 3$$

Let's now look at how to simplify exponential expressions that involve addition and subtraction.

Example 3

Simplify: $\dfrac{9^x - 3 \cdot 3^{2x-3}}{3 \cdot 9^{x-1}}$

Solution

Our first step is again to write all our bases as prime numbers.

$$\frac{3^{2x} - 3 \cdot 3^{2x-3}}{3 \cdot 3^{2x-2}}$$

Because of the negative (–) sign in the numerator we **cannot** simplify this expression in the same manner as the previous example.

We need to factorise: $\dfrac{\cancel{3^{2x}}(1 - 3 \cdot 3^{-3})}{3 \cdot \cancel{3^{2x}} 3^{-2}}$

Now we may cancel the 3^{2x} in the numerator and denominator.

$$\frac{(1 - 3^{-2})}{3^{-1}}$$

$$= \frac{1 - \frac{1}{9}}{\frac{1}{3}}$$

$$= \frac{8}{3}$$

Let's have a look at another example.

Example 4

Simplify: $\dfrac{3 \cdot 2^{x+1} - 4 \cdot 2^{x-1}}{2^{x-3}}$

Solution

In this example we can take out 2^x as a common factor.

$$\frac{2^x(3 \cdot 2^1 - 4 \cdot 2^{-1})}{2^x \cdot 2^{-3}}$$

$$= \frac{6 - 2}{\frac{1}{8}}$$

$$= 32$$

Addition, subtraction and multiplication of surds

Working with surds has become easy in recent times. Thanks to advanced calculators that are able to give answers in surd form.

It is still important however to understand basic operations.

Here are some important facts to remember.

- $\sqrt{x} = x^{\frac{1}{2}}$
- $\sqrt[3]{x} = x^{\frac{1}{3}}$
- $\sqrt[n]{x} = x^{\frac{1}{n}}$

Let's have a look at how to multiply surds.

Example

Example 1

Simplify: $\sqrt{15} \times \sqrt{12}$

Solution

Solution

Again as with exponents we write the surds as products of prime number.

$\sqrt{5 \times 3} \times \sqrt{2^2 \times 3}$

$= \sqrt{5} \times \sqrt{3} \times 2\sqrt{3}$

As $\sqrt{}$ and $(\ \ )^2$ are inverse operations they cancel each other out $\sqrt{2^2} = 2$

Also remember that $\sqrt{x} \times \sqrt{x} = x$

$\therefore \sqrt{5} \times \sqrt{3} \times 2\sqrt{3}$

$= \sqrt{5} \times 2 \times 3$

$= 6\sqrt{5}$

Let's have a look at an example of addition and subtraction.

Example

Example 2

Simplify: $\sqrt{20} + \sqrt{45}$

Solution

Solution

Again we reduce 20 and 45 to products of their prime numbers.

$\sqrt{2^2 \times 5} + \sqrt{3^2 \times 5}$

$= 2\sqrt{5} + 3\sqrt{5}$

$= 5\sqrt{5}$

We add $2\sqrt{5}$ and $3\sqrt{5}$ together.

You should now try the following exercise to test your skills on exponents and surds.

Activity

Activity 4

Simplify:

1. $\dfrac{9^{a-2}.10^{a-2}}{6^{a-4}.15^{a}}$

Simplify:

2. $\dfrac{\sqrt{4^{x+1}} \times \sqrt[3]{8^{x+1}}}{4^{x+1}}$

....

....

Simplify:

3. $\left(27^{\frac{2}{3}} + 9^{\frac{3}{2}}\right)81^{\frac{-3}{4}}$

Simplify:

4. $3\sqrt{45} + 2\sqrt{80} - 2\sqrt{125}$

Simplify:

5. $\left[\dfrac{16x^{\frac{-5}{6}}}{81\sqrt{x}}\right]^{\frac{-3}{4}}$

Simplify:

6. $\dfrac{5^{2k+1}}{5^{2k-1}} - \dfrac{5^{2k-1}}{(5^k)^2}$

Simplify:

7. $\dfrac{2^x - 2^{-x}}{4^x - 1}$

8. A rectangle has an area of 1. If the length is $(\sqrt{7} + 6)$ cm long. Determine the width in the form $a\sqrt{b} + c$

9. Solve for x leaving your answer in surd form.
$\dfrac{\sqrt{20x} - \sqrt{5x}}{1 - \sqrt{3}} = 1$

Solutions to Activities

Activity 1

1. $x^2 - 12x + 34$

   $= x^2 - 12x + 6^2 - 6^2 + 34$

   $= (x - 6)^2 - 2$

2. $a^2 - 8a + 15$

   $= a^2 - 8a + 4^2 - 4^2 + 15$

   $= (a - 4)^2 - 1$

3. $-2b^2 - b - 4 = -2(6^2 + \frac{6}{2} + 2)$

   $-2\left(b^2 + \frac{1}{2}b + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2\right)$

   $= -2\left[\left(b + \frac{1}{4}\right)^2 + \frac{-1}{16} + 2\right]$

   $= -2\left[\left(b + \frac{1}{4}\right)^2 + \frac{31}{16}\right]$

   $= -2\left(b + \frac{1}{4}\right)^2 - \frac{31}{8}$

Activity 2

1. $\frac{x^2 - x - 12}{12 - 3x}$

   $= \frac{(x - 4)(x + 3)}{3(4 - x)}$

   $= \frac{\cancel{(4 - x)}(x + 3)}{3\cancel{(4 - x)}}$

   $= -\frac{(x + 3)}{3}$

2. Take $(x - 2)$ out as a common factor.

   $\frac{(x - 2)[(2x - 4) - (3x + 4)]}{-(x - 2)(x + 8)}$

   $= \frac{(x - 2)[2x - 4 - 3x - 4]}{-(x - 2)(x + 8)}$

   $= \frac{-(x + 8)}{-(x + 8)}$

   $= 1$

3. $\frac{a^2}{{}_2\cancel{16}\cancel{(a - 1)}} \times \frac{\cancel{9}a^4}{4} \times \frac{\cancel{8}\cancel{(a - 1)}(a - 1)}{\cancel{9}}$

   $= \frac{a^6(a - 1)}{2 \times 4}$

   $= \frac{a^6(a - 1)}{8}$

Activity 3

1. $\frac{2(x + 3) - 1}{(x + 3)^2}$ LCD $= (x + 3)^2$

   $= \frac{2x + 6 - 1}{(x + 3)^2}$

   $= \frac{2x + 5}{(x + 3)^2}$

2. $\frac{5}{3} + \frac{2}{x - 2} + \frac{3}{2} - \frac{3}{2(x - 2)}$ LCD $= 6(x - 2)$

   $= \frac{5 \times 2(x - 2) + 2 \times 6 + 3 \times 3(x - 2) - 3 \times 3}{6(x - 2)}$

   $= \frac{10x - 20 + 12 + 9x - 18 - 9}{6(x - 2)}$

   $= \frac{19x - 35}{6(x - 2)}$

3. $\frac{3}{x + 2} - \frac{2}{3(x - 1)} + \frac{1}{x - 1}$ LCD $= 3(x + 2)x - 1)$

   $= \frac{3 \times 3(x - 1) - 2(x + 2) + 1 \times 3(x + 2)}{3(x + 2)(x - 1)}$

   $= \frac{9x - 9 - 2x - 4 + 3x + 6}{3(x + 2)(x - 1)}$

   $= \frac{10x - 7}{3(x + 2)(x - 1)}$

Activity 4

1. $\dfrac{9^{a-2}.10^{a-2}}{6^{a-4}.15^{a}}$

$= \dfrac{(3^2)^{a-2}.(2.5)^{a-2}}{(2.3)^{a-4}.(3.5)^{a}}$

$= \dfrac{3^{2a-4}.2^{a-2}.5^{a-2}}{2^{a-4}.3^{a-4}.3^{a}.5^{a}}$

$= 2^{a-2-a+4}.3^{2a-4-a+4-a}.5^{a-2-a}$

$= 2^2.3^0.5^{-2}$

$= \dfrac{4}{25}$

2. $\dfrac{\sqrt{4^{x+1}} \times \sqrt[3]{8^{x+1}}}{4^{x+1}}$

$= \dfrac{(2^{2x+2})^{\frac{1}{2}} \times (2^{3x+3})^{\frac{1}{3}}}{2^{2x+2}}$

$= \dfrac{2^{x+1} \times 2^{x+1}}{2^{2x+2}}$

$= \dfrac{2^{2x+2}}{2^{2x+2}} = 1$

3. $(27^{\frac{2}{3}} + 9^{\frac{3}{2}}).81^{-\frac{3}{4}}$

$= ((3^3)^{\frac{2}{3}} + (3^2)^{\frac{3}{2}}).(3^4)^{-\frac{3}{4}}$

$= (3^2 + 3^3).3^{-3}$

$= \dfrac{9+27}{27}$

$= \dfrac{36}{27} = \dfrac{4}{3}$

4. $3\sqrt{45} + 2\sqrt{80} - 2\sqrt{125}$

$= 3\sqrt{3^2 \times 5} + 2\sqrt{2^4 \times 5} - 2\sqrt{5^3}$

$= 9\sqrt{5} + 8\sqrt{5} - 10\sqrt{5}$

$= 7\sqrt{5}$

5. $\left[\dfrac{16x^{\frac{-5}{6}}}{81\sqrt{x}}\right]^{\frac{-3}{4}}$

$= \left[\dfrac{2^4.x^{-\frac{5}{6}}}{3^4x^{\frac{1}{2}}}\right]^{-\frac{3}{4}}$

$= \dfrac{2^{-3}.x^{\frac{5}{8}}}{3^{-3}x^{-\frac{3}{8}}}$

$= \dfrac{27x^{\frac{5}{8}+\frac{3}{8}}}{8}$

$= \dfrac{27x}{8}$

6. $\dfrac{5^{2k+1}}{5^{2k-1}} - \dfrac{5^{2k-1}}{(5^k)^2}$

$= 5^{2k+1-2k+1} - 5^{2k-1-2k}$

$= 5^2 - \dfrac{1}{5}$

$= 25 - \dfrac{1}{5}$

$= 24\frac{4}{5}$

7. $\dfrac{2^x - 2^{-x}}{2^{2x} - 1}$

$= \dfrac{2^x - \frac{1}{2^x}}{2^{2x} - 1}$

$= \dfrac{2^{2x} - 1}{2^x} \times \dfrac{1}{2^{2x} - 1}$

$= \dfrac{1}{2^x}$

8. length $= (\sqrt{7} + 6)$

$\therefore$ $(\sqrt{7} + 6)$. width $= 1$

$\therefore$ width $= \dfrac{1}{(\sqrt{7} + 6)}$

rationalise: $\dfrac{1}{(\sqrt{7}+6)} \times \dfrac{\sqrt{7}-6}{\sqrt{7}-6}$

$= \dfrac{\sqrt{7}-6}{7-36}$

$= \dfrac{\sqrt{7}-6}{29}$

$= \dfrac{-1}{29}\sqrt{7} + \dfrac{6}{29}$

9. $\dfrac{\sqrt{20x} - \sqrt{5x}}{1 - \sqrt{3}} = 1$

$\dfrac{2\sqrt{5x} - \sqrt{x}}{1 - \sqrt{3}} = 1$

$\dfrac{\sqrt{5x}}{1 - \sqrt{3}} = 1$

$\sqrt{5x} = 1 - \sqrt{3}$

$\sqrt{5}\, x^{\frac{1}{2}} = 1 - \sqrt{3}$

$x^{\frac{1}{2}} = \dfrac{1 - \sqrt{3}}{\sqrt{5}}$

$x = \left[\dfrac{1 - \sqrt{3}}{\sqrt{5}}\right]^2$

$x = \dfrac{1 - 2\sqrt{3} + 3}{5}$

$= \dfrac{4 - 2\sqrt{3}}{\sqrt{5}}$